The Space of Continuous Functions

Note Title 7/18/2015

Given a metric space (M, d) we denote by $C(M)$ the space of all continuous functions $f : M \to \mathbb{R}$, where we use the usual (absolute value) metric on $\mathbb{R}$. Just as in a first-year calculus course, once we have a continuous function we want to know how we can use it to build other continuous functions, and similarly how we can recognize that a combination of functions is continuous. We already know that the composition of continuous functions is continuous, and now we'll consider other ways of combining continuous functions.

Lemma 5.8

If $f, g : M \to \mathbb{R}$ are continuous, then so is the function $h : M \to \mathbb{R}^2$, defined by $h(x) = (f(x), g(x))$ for $x \in M$, where we use the Euclidean norm on $\mathbb{R}^2$.

Proof

Take $x_n \to x$ in M, and note by continuity that $f(x_n) \to f(x)$ and $g(x_n) \to g(x)$ in $\mathbb{R}$. We've established that $(f(x_n), g(x_n))$ converges in $\mathbb{R}^2$ iff the components converge individually, so $(f(x_n), g(x_n)) \to (f(x), g(x))$.

$\square$

Theorem 5.10

Let $f, g : M \to \mathbb{R}$ be continuous. Then $f \pm g$, fg, $\max\{f, g\}$, and $\min\{f, g\}$ are all continuous.

Proof

We'll prove the claim for $f+g$. The others are similar. Since we already know that the composition of two continuous functions is continuous, the idea of the proof is to express these combinations as compositions.

For $f(x)+g(x)$, we compose $\phi(x,y) = x+y$ with $h(x) = (f(x), g(x))$.

I.e.,

$$\phi \circ h(x) = f(x) + g(x)$$

Now we've already checked that h is continuous (Lemma 5.8), so we only need to check that $\phi(x,y) = x+y$ is continuous.

But this is trivial, because

$$|\phi(x,y) - \phi(x_0, y_0)| = |x - x_0 + y - y_0|$$
$$\leq |x - x_0| + |y - y_0|$$

So if $x_n \to x_0$ and $y_n \to y_0$ then

$$\phi(x_n, y_n) \longrightarrow \phi(x_0, y_0),$$

and this means that ϕ is continuous. $\square$