Note Title 7/18/2015

Algebras, Groups, Rings, and Lattices

Definitions

(i) An algebra is a vector space equipped with a product. I.e., $C(M)$ is an algebra (in addition to being a vector space) because we can make sense of fg for $f, g \in C(M)$. (By the way $fg(x)$ is exactly what you expect: $f(x)g(x)$.)

(ii) A <u>group</u> (which we'll need in defining a ring) is a collection of objects defined with an operation that we'll think of as addition. A group $\mathcal{G}$ has the following four properties:

1. If $x, y \in \mathcal{G}$ then $x + y \in \mathcal{G}$
2. If $x, y, z \in \mathcal{G}$ then $(x + y) + z = x + (y + z)$

(i.e., associativity)

3. There is a zero element $0 \in \mathcal{G}$ so that $x + 0 = x$ for all $x \in \mathcal{G}$.

4. For every $x \in \mathcal{G}$ there is an inverse element y so that $x + y = 0$. (I.e., $y = -x$.)

One common example of a group is the integers $\mathbb{Z}$.

(iii) A <u>ring</u> is an Abelian (i.e., commutative) group equipped with a product that is associative and is distributive over addition. (The operations I've called addition and multiplication may be generalized.)

(iv) A <u>lattice</u> is a partially ordered set in which every two elements has a supremum and infimum. (A partially ordered set is a collection of elements with an order relation "$\leq$" that is reflexive ($a \leq a$ for every element of the set), antisymmetric ($a \leq b$ and $b \leq a \Rightarrow a = b$), and transitive ($a \leq b$ and $b \leq c \Rightarrow a \leq c$). For a partially ordered set we don't require either $a \leq b$ or $b \leq a$.) $C(M)$ is a lattice because we can make sense of

$\max\{f, g\}$ and $\min\{f, g\}$.