Lebesgue's Space-Filling Curve

Note Title 7/23/2015

A second approach to constructing a space-filling curve from $[0,1]$ to $[0,1]\times[0,1]$ is due to Lebesgue.

Lebesgue's idea is that since the Cantor set Δ can be mapped onto all of $[0,1]$ (by the Cantor function) it might be possible to modify the Cantor function so that it maps Δ onto $[0,1]\times[0,1]$.

And in fact this is possible.

To see how this works, recall that each $t \in \Delta$ can be expressed as

$$t = \sum_{n=1}^{\infty} \frac{2a_n}{3^n}$$

where each a_n is either 0 or 1. Lebesgue's map is:

$$f(t) = (x(t), y(t))$$

where

$$x(t) = \sum_{n=1}^{\infty} \frac{a_{2n}}{2^n}$$

and

$$y(t) = \sum_{n=1}^{\infty} \frac{a_{2n-1}}{2^n}.$$

As in our proof of Corollary 2.16, we can show that $x(t)$ and $y(t)$ both map onto $[0,1]$.

Just as we extended the Cantor function to a continuous function, we can extend $x(t)$ and $y(t)$ to be continuous on $[0,1]$. In this way, we get a continuous space-filling curve $f:[0,1] \longrightarrow [0,1] \times [0,1]$.

In fact, we can use this method to construct space-filling curves from $[0,1]$ to $[0,1]\times[0,1]\times[0,1]$ (i.e. $[0,1]^3$), $[0,1]^4$ etc.

For example, for $[0,1]^3$ we take

$$f(t) = (x(t), y(t), z(t))$$

and set

$$x(t) = \sum_{n=1}^{\infty} \frac{a_{3n}}{2^n}$$

$$y(t) = \sum_{n=1}^{\infty} \frac{a_{3n-1}}{2^n}$$

$$z(t) = \sum_{n=1}^{\infty} \frac{a_{3n-2}}{2^n}.$$