Precise Definition of Connectedness

Note Title 7/19/2015

Definition

We say that a metric space (M,d) is **disconnected** if there exist non-empty open sets A and B so that $A \cap B = \emptyset$ and $A \cup B = M$. Otherwise, we say (M,d) is connected.

Our first example was $M = [a,b]$, which we showed is connected.

In fact, we could just as well state this definition in terms of closed sets. That is, we know A and B are both complements of open sets ($A^c = B$, $B^c = A$), and so A and B are closed. (That is, they are clopen.) We see that if M is disconnected it must have non-trivial clopen sets (i.e., clopen sets other than $\emptyset$ and M).

On the other hand, suppose M has a non-trivial clopen set. Then A^c is open and

$$A \cap A^c = \emptyset \quad \text{and} \quad A \cup A^c = M,$$

so M is disconnected. This provides us with a theorem:

Theorem 6.1

M is connected iff M contains no non-trivial clopen sets.