Note Title 7/19/2015

Disconnections

So far we've been interested in whether or not a full space M is connected, but we know properties such as open and closed can be different for subsets. For example, the set $(0,1)$ is open as a subset of $\mathbb{R}$ (and not closed), but $M = (0,1)$ as a full space is both open and closed.

In order to investigate connectedness for subsets, let's suppose $E \subset M$ is disconnected, meaning that there exist disjoint, non-empty sets U and V that are open in E so that $E = U \cup V$.

We know from Proposition 4.13 that there exist open sets $A, B \subset M$ (open in M) so that $U = A \cap E$ and $V = B \cap E$.

We see that

$$E = U \cup V = (A \cap E) \cup (B \cap E) \subset A \cup B.$$

We would like to check if the sets A and B can be taken as disjoint. They can, as we'll state in the next lemma.

Lemma 6.3

Let E be a subset of a metric space M. If U and V are disjoint open sets in E, then there are disjoint open sets A and B in M so that $U = A \cap E$ and $V = B \cap E$.

We see from this lemma that E is disconnected iff there exist disjoint, non-empty sets A and B, open in M, so that:

$A \cap E \neq \emptyset$, $B \cap E \neq \emptyset$, and $E \subset A \cup B$.

Notice that if $E = M$, this says:

$A \neq \emptyset$, $B \neq \emptyset$, and $M = A \cup B$

(We have: $M \subset A \cup B \subset M$.) We can take the following definition as a generalization to subsets of our definition of a disconnected space:

Definition

We say that a subset E of a metric space M is *disconnected* if there exists a pair of disjoint open sets A and B so that $A \cap E \neq \emptyset$, $B \cap E \neq \emptyset$, and $E \subset A \cup B$. We refer to the sets A and B as a *disconnection* of E.