Lemma 6.3

Let E be a subset of a metric space M. If U and V are disjoint open sets in E, then there are disjoint open sets A and B in M so that $U = A \cap E$ and $V = B \cap E$.

Proof

One step of the proof will be carried out in

Problem 6.1.

For each $x \in U$ there is an $\varepsilon_x > 0$ so that

$$E \cap B_{\varepsilon_x}(x) \subset U$$

(since U is open in E), and likewise for each $y \in V$ there is a $\delta_y > 0$ so that

$$E \cap B_{\delta_y}(y) \subset V.$$

Since $U \cap V = \emptyset$ it's clear that

$$(E \cap B_{\varepsilon_x}(x)) \cap (E \cap B_{\delta_y}(y)) = \emptyset,$$

which is equivalent to

$$E \cap B_{\varepsilon_x}(x) \cap B_{\delta_y}(y) = \emptyset.$$

<u>Claim</u> We also must have

$$B_{\frac{\varepsilon_x}{2}}(x) \cap B_{\frac{\delta_y}{2}}(y) = \emptyset \quad \forall\, x \in U,\, y \in V.$$

(This will be verified in Problem 6.1.)

Now set

$$A := \bigcup \left\{ B_{\frac{\varepsilon_x}{2}}(x) : x \in \mathcal{U} \right\}$$

$$B := \bigcup \left\{ B_{\frac{\delta_y}{2}}(y) : y \in V \right\}.$$

Then A and B are open subsets of M, disjoint by the claim. Also, $\mathcal{U} \subset A$ and $V \subset B$.

We have:

$\mathcal{U} \subset A$ and $\mathcal{U} \subset E$, so $\mathcal{U} \subset A \cap E$, but also $B_{\frac{\varepsilon_x}{2}}(x) \cap E \subset \mathcal{U}$ $\forall x \in \mathcal{U}$, so $A \cap E \subset \mathcal{U}$. We see that $\mathcal{U} = A \cap E$, and likewise $V = B \cap E$. $\square$