Note Title 7/20/2015

Connected Subsets of $\mathbb{R}$

Theorem 6.4

A subset E of $\mathbb{R}$, containing more than one point, is connected iff whenever $x, y \in E$ with $x < y$ we also have $[x, y] \subset E$. That is, the connected subsets of $\mathbb{R}$ (containing more than one point) are precisely the intervals.

Proof

($\Longrightarrow$) First, suppose $E \subset \mathbb{R}$ is connected and contains at least two points, and suppose $x, y \in E$ with $x < y$. We need to show $[x, y] \subset E$. Suppose not. Then there exists $z \in (x, y)$ so that $z \notin E$. Then certainly

$$E \subset (-\infty, z) \cup (z, \infty),$$

which means precisely that E is disconnected

with disconnection $A = (-\infty, z)$, $B = (z, \infty)$.
This is a contradiction.

($\Leftarrow$) For the other way, we assume that for any $x, y \in E$ with $x < y$ we have $[x,y] \subset E$. We need to show that E is connected, which is to say that it is <u>not</u> disconnected. Again, we argue by contradiction, assuming E <u>is</u> disconnected.

In this case there exist disjoint open sets A and B so that

$$A \cap E \neq \emptyset, \quad B \cap E \neq \emptyset, \quad \text{and } E \subset A \cup B.$$

Take $a \in A \cap E$ and $b \in B \cap E$, and notice that since $a \neq b$ one must be larger. We'll take the case $a < b$, so that $[a,b] \subset E$.

We have, then,

$$[a,b] \subset E \subset A \cup B,$$

which says that A and B are a disconnection for $[a,b]$, but we've already seen that intervals of the form $[a,b]$ are connected, and this is a contradiction.

Last, we prove the claim in the second sentence of our statement that if $[x,y] \subset E$ whenever $x,y \in E$ with $x<y$ then E is an interval. Clearly, E must contain the open interval

$(\inf E, \sup E)$. (Just take a sequence (b_n) so that $b_n \to \sup E$, so that (proceeding similarly for (a_n)) $[a_n, b_n] \subset E$ for all n.)

We have, then

$$(\inf E, \sup E) \subset E \quad \text{and} \quad E \subset [\inf E, \sup E].$$

We see that E must have one of 4 possible forms: $(\inf E, \sup E)$, $(\inf E, \sup E]$, $[\inf E, \sup E)$, $[\inf E, \sup E]$. $\square$