If A and B are Connected, $A \times B$ is Connected

Note Title 7/21/2015

Lemma 6.9

If A and B are connected, then $A \times B$ is connected.

Proof

Let f be any continuous function $f : A \times B \to \{0,1\}$.

To see that $A \times B$ is connected, we need (by Lemma 6.5) to see that f cannot map onto $\{0,1\}$, which means we need to show that f is identically constant: always 0 or always 1.

To see this, let $a \in A$ and $b' \in B$, and note that each of the functions $f(a,\cdot): B \to \{0,1\}$ and $f(\cdot, b'): A \to \{0,1\}$ is continuous. But A and B are both connected so each of these maps must be constant.

So for each $a \in A$

$$* \qquad f(a,b) = \text{constant} \quad \forall\, b \in B$$

and likewise for each $b' \in B$

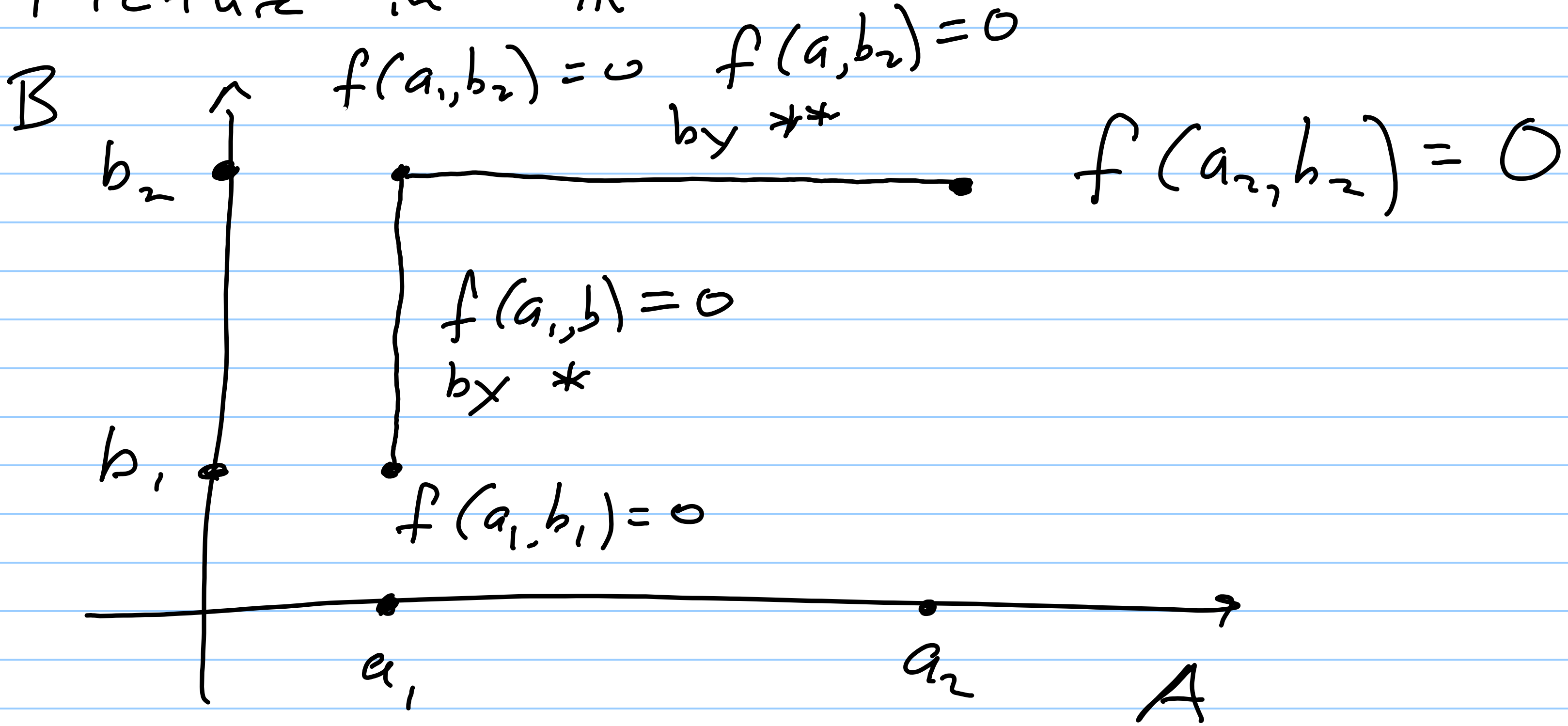
$$f(a,b') = \text{constant} \quad \forall\, a \in A.$$

Suppose $f(a_1,b_1) = 0$ for some $(a_1,b_1) \in A \times B$.

We want to show that for any $(a_2,b_2) \in A \times B$

we have $f(a_2,b_2) = 0$. Then f must be constant, and we're finished.

Picture in $\mathbb{R}^2$:



$\square$