Is $\mathbb{R}$ Homeomorphic to $\mathbb{R}^2$?

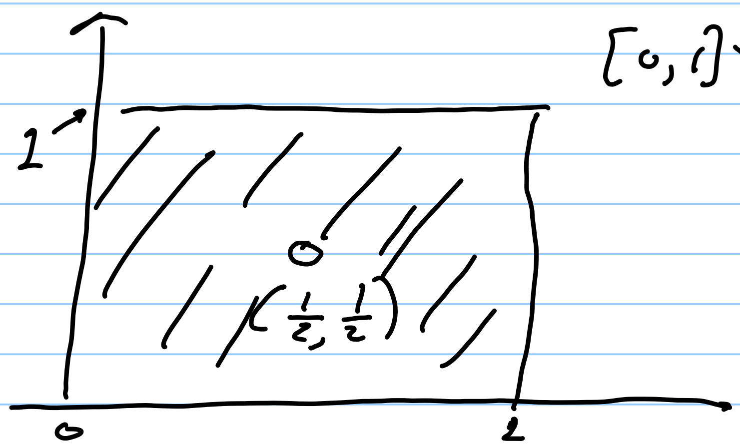
Note Title 7/22/2015

Let's start by showing that $[0,1]$ is not homeomorphic to $[0,1] \times [0,1]$. First, we know that $[0,1]$ is connected, and we also know from lemma 6.9 that $[0,1] \times [0,1]$ is connected. Suppose $[0,1]$ is homeomorphic to $[0,1] \times [0,1]$, meaning that there exists a homeomorphism

$$f : [0,1] \to [0,1] \times [0,1].$$

If we delete a point from $[0,1]\times[0,1]$ we can use the proof of Lemma 6.9 to show that the resulting set is still connected. For example, we can remove $\{(\frac{1}{2},\frac{1}{2})\}$ to get $[0,1]\times[0,1] \setminus \{(\frac{1}{2},\frac{1}{2})\}$:

$[0,1]\times[0,1]$



Now remove any value $c \in (0,1)$ from $[0,1]$, and notice that f would be a homeomorphism from $[0,1] \setminus \{c\} = [0,c) \cup (c,1]$ (a disconnected set) to $[0,1] \times [0,1] \setminus \{f(c)\}$ (a connected set). This contradicts the fact that a continuous function maps connected sets to connected sets.

Since $\mathbb{R}^2$ can be regarded as $\mathbb{R}\times\mathbb{R}$ we know $\mathbb{R}^2$ is connected (since $\mathbb{R}$ is, by Lemma 6.4), and the same argument as above (using the fact that $\mathbb{R}^2 \setminus \{p\}$ is

individual point

connected) shows that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$. Likewise, we can similarly show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for any $n = 2, 3, \ldots$.

We run into a problem when we consider $\mathbb{R}^n$ and $\mathbb{R}^m$, $n, m = 2, 3, \ldots$, $n \neq m$. Each of these will be connected when we delete a point, so the argument above won't work in these cases. It's a bit difficult to show, but $\mathbb{R}^n$ is <u>not</u> homeomorphic to $\mathbb{R}^m$ in these cases.