Alternative Formulation of Total Boundedness

Note Title 7/24/2015

In our definition of total boundedness, we took the points $\{x_i\}_{i=1}^{n} \subset M$, but we could just as well have taken them in A. To see this, notice that given any $\varepsilon > 0$ we can find $\{x_i\}_{i=1}^{n} \subset M$ so that

$$A \subset \bigcup_{i=1}^{n} B_{\frac{\varepsilon}{2}}(x_i).$$

We can assume $A \cap B_{\frac{\varepsilon}{2}}(x_i) \neq \emptyset \ \forall i$,

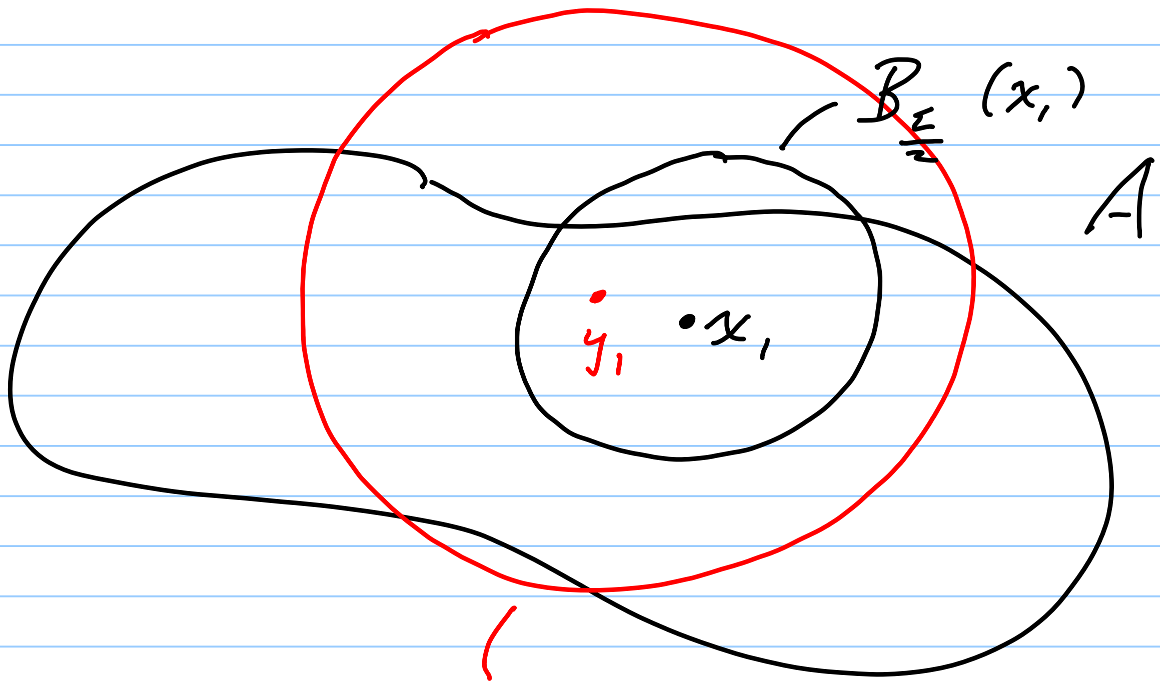
because otherwise we could throw out x_i.

For each i, we can choose $y_i \in A \cap B_{\frac{\varepsilon}{2}}(x_i)$,

and clearly (no matter how we choose these y_i)

$$A \subset \bigcup_{i=1}^{n} B_{\varepsilon}(y_i).$$

Picture:



$$B_{\frac{\varepsilon}{2}}(x_1) \subset B_{\varepsilon}(y_1)$$

We can even be a bit more general:

Lemma 7.1

A is totally bounded iff given any $\varepsilon > 0$ there are finitely many sets $A_1, A_2, \dots, A_n \subset A$ with $\operatorname{diam}(A_i) < \varepsilon$ $\forall i$ so that $A \subset \bigcup_{i=1}^{n} A_i$.

Recall from Chapter 3 that

$$\operatorname{diam}(A) := \sup \{ d(a,b) : a, b \in A \}.$$

Proof

For ($\Rightarrow$) suppose A is totally bounded, and let $\varepsilon > 0$ be given. By definition we can take $\{x_i\}_{i=1}^{n} \subset M$ so that

$$A \subset \bigcup_{i=1}^{n} B_{\frac{\varepsilon}{2}}(x_i).$$

Then take $A_i := A \cap B_{\frac{\varepsilon}{2}}(x_i)$ to get the required sets $A_1, A_2, \ldots, A_n$.

For ($\Leftarrow$) let $\varepsilon > 0$ be given, and let $A_1, A_2, \ldots, A_n$ be as described. For each A_i, take $x_i \in A_i$ and notice that

$$A_i \subset B_{2\varepsilon}(x_i)$$

(because $\operatorname{diam}(A_i) < \varepsilon$). But then

$$A \subset \bigcup_{i=1}^{n} B_{2\varepsilon}(x_i),$$

and this means that A is totally bounded. $\square$