Note Title 7/25/2015

Cauchy Sequences and Total Boundedness, I

Lemma 7.3

Let $(x_n) \subset (M,d)$, and let $A = \{x_n : n \geq 1\}$.

(i) If (x_n) is Cauchy then A is totally bounded.

(ii) If A is totally bounded, then (x_n) has a Cauchy subsequence.

Proof

For (i), assume (x_n) is Cauchy and let $\varepsilon > 0$ be given. By the Cauchy property there exists an integer N so that

$$m, n > N \implies d(x_m, x_n) < \frac{\varepsilon}{2}$$

$$\implies \operatorname{diam}\{x_n : n > N\} \le \frac{\varepsilon}{2}.$$

We can write:

$$A = \{x_1\} \cup \{x_2\} \cup \dots \cup \{x_N\} \cup \{x_n : n > N\}$$

We can cover $\{x_n : n > N\}$ by a set of diameter ε, because $\operatorname{diam}\{x_n : n > N\} \leq \frac{\varepsilon}{2}$. The other sets are individual points, so we can cover them with balls $B_{\frac{\varepsilon}{2}}(x_i)$, for which $\operatorname{diam}(B_{\frac{\varepsilon}{2}}(x_i)) < \varepsilon$. In this way we've covered A by a finite collection of sets $A_i \subset A$ with $\operatorname{diam}(A_i) < \varepsilon$, and we can conclude that A is totally bounded.

For (iii), we first note that if A is finite (i.e. we only have a finite collection of distinct points) then some element must be repeated an infinite number of times, and this gives a trivial Cauchy subsequence. In this way, we can assume A has an infinite number of distinct elements.

Since A is totally bounded we can cover it with a finite collection of sets with diameters less than ε for any $\varepsilon > 0$, so in particular we can take $\varepsilon = 1$. Now since A has an infinite number of elements, and our cover only has a finite number of sets, there must be at least one set in our cover that covers an infinite number of elements. Call this set A_1, and keep in mind $A_1 \subset A$. Since A is totally bounded, A_1 is totally bounded, and so A_1 can be covered by a finite collection of sets, and we take these sets to have diameters less than $1/2$. At least one of these sets must cover an infinite number of points, and we denote this set A_2, keeping in mind that $A_2 \subset A_1$. Continuing this process we obtain a sequence of sets

$$A \supset A_1 \supset A_2 \supset A_3 \supset \dots,$$

where each A_k contains infinitely many x_n, and $\operatorname{diam}(A_k) < 1/k$.

We now construct a subsequence by taking $x_{n_1} \in A_1$, $x_{n_2} \in A_2 \setminus \{x_{n_1}\}$, $x_{n_3} \in A_3 \setminus \{x_{n_1}, x_{n_2}\}$ etc.

But this subsequence (x_{n_j}) is Cauchy because

$$\{x_{n_j} : j \geq K\} \subset A_K$$

which means

$$\operatorname{diam}\{x_{n_j} : j \geq K\} < \frac{1}{K},$$

which means

$$d(x_{n_j}, x_{n_i}) < \frac{1}{K}$$

for n_i, n_j sufficiently large. This means precisely that (x_{n_j}) is Cauchy, and so we've established that we have a Cauchy subsequence. $\square$