Note Title 7/25/2015

Cauchy Sequences and Total Boundedness, II

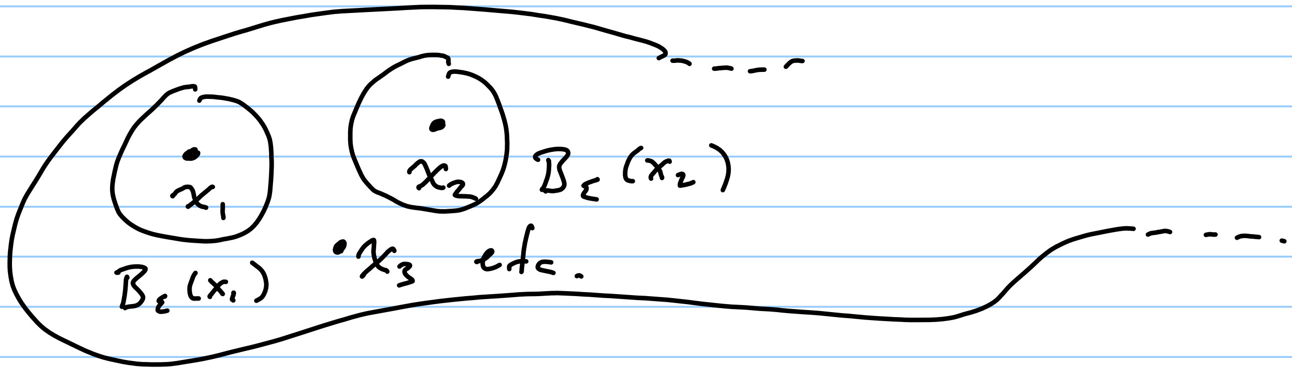
Theorem 7.5

A set A is totally bounded iff every sequence in A has a Cauchy subsequence.

Proof

First, ($\Rightarrow$) is part (ii) of Lemma 7.3.

For ($\Leftarrow$) suppose every sequence in A has a Cauchy subsequence, but that A is not totally bounded. Then there is some $\varepsilon > 0$ so that A cannot be covered by a finite number of balls with radius ε. We construct a sequence as follows: let $x_1 \in A$, and find $x_2 \in A$ so that $d(x_1, x_2) \geq \varepsilon$. (If this isn't possible then A can be covered by $B_\varepsilon(x_1)$.) Next, take $x_3 \in A$ so that $d(x_1, x_3) \geq \varepsilon$, $d(x_2, x_3) \geq \varepsilon$. (If this isn't possible, then A can be covered by $B_\varepsilon(x_1)$ and $B_\varepsilon(x_2)$.)



Continuing in this way, we construct a sequence (x_n) so that $d(x_m, x_n) \geq \varepsilon$ $\forall\ n \neq m$. But this sequence cannot have a Cauchy subsequence, so we get a contradiction to the assumption that A is not totally bounded. We can conclude that A is totally bounded. $\square$

Corollary 7.6 (Alternative Statement of the Bolzano - Weierstrass Theorem)

Every bounded infinite subset of $\mathbb{R}$ has a limit point in $\mathbb{R}$.

Proof

The point here is simply that there is a bounded infinite sequence (of distinct elements)

(x_n) in any such subset, and in $\mathbb{R}$ the subset is totally bounded, so the sequence has a Cauchy subsequence by Theorem 7.5. But Cauchy sequences converge in $\mathbb{R}$, so this subsequence converges to some $x \in \mathbb{R}$. $\square$