MATH 609-600, Exam #2 – Solutions
November 18, 2014

(1) (20 pts) Determine the nodes and weights for the formula with highest degree of accuracy

$$\int_{-1}^{1} x^2 f(x)\, dx \approx A_0 f(x_0) + A_1 f(x_1).$$

What is the degree of accuracy of this rule?

Solution: This is the two pout Gaussian rule for the weight $w(x) = x^2$. The degree of accuracy (DAC) must be 3. Because of the symmetry the coefficients are the same and the points are symmetric: $A_0 = A_1$ and $1 > x_1 = -x_0 > 0$. Using that the rule is exact for 1 and x^2 we find: $A_0 = A_1 = \frac{1}{3}$ and $x_1 = -x_0 = \sqrt{\frac{3}{5}}$.

(2) (20 pts) Prove that all coefficients of the Gaussian rule $\int_{-1}^{1} f(x)\, dx \approx \sum_{i=0}^{n} A_i f(x_i)$ are positive.

Solution: The Gaussian rule is exact for all polynomials of degree $2n+1$. Fix an index j, $0 \le j \le n$. Then the Gaussian rule will be exact for the function $f_j(x) = \prod_{i \ne j} \left(\frac{x - x_i}{x_j - x_i} \right)^2$.

The rule applied for f_j reduces to

$$0 < \int_{-1}^{1} \prod_{i \ne j} \left(\frac{x - x_i}{x_j - x_i} \right)^2 dx = \sum_{i=0}^{n} A_i f_j(x_i) = A_j.$$

(3) (20 pts) Use Lagrange interpolation to derive the formula for numerical differentiation of $f'(x)$ and the approximation error of the rule using the values $f(x)$, $f(x+h)$, and $f(x+4h)$.

Solution: Using Taylor series we obtain

$$f'(x) = \frac{-15f(x) + 16f(x+h) - f(x+2h)}{12h} + \mathcal{O}(h^2).$$

(4) (20 pts) Prove that

$$f[0, 1, \ldots, m] = \frac{1}{m!} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(j).$$

Solution: In class (and in HW) we showed that $f[0, 1, \ldots, m] = \sum_{j=0}^{m} f(j) \prod_{i \neq j} \frac{1}{j-i}$ and the problem follows by observing that $\prod_{i \neq j} \frac{1}{j-i} = \frac{1}{m!}(-1)^{m-j}\binom{m}{j}$.

(5) (20 pts) Let $f(x)$ has the following values: $f(0) = 1$, $f'(0) = 1$, $f''(0) = 0$, $f(1) = 2$, and $f'''(1) = 2014$. Find a polynomial $p(x)$ of minimal degree interpolating this data, i.e., $f(0) = p(0)$, $f(1) = p(1)$, $f^{(j)}(0) = p^{(j)}(0)$ for $j = 1, 2$, and $f^{(3)}(1) = p^{(3)}(1)$.

Solution: Using the divided difference formula (or Hermite formula) we derive that $p(x) = 1+x$ is the unique cubic polynomial which interpolates the data $f(0) = 1$, $f'(0) = 1$, $f''(0) = 0$, and $f(1) = 2$. Unfortunately $p'''(1) \neq 2014$ and we need to find a fourth degree polynomial to fit this data set. The way to get one is to seek a function q such that:

$$q(x) = p(x) + \alpha x^3(x-1)$$

and $q'''(1) = 2014$. Solving this equation for α gives

$$q(x) = p(x) + \frac{1007}{9} x^3(x-1).$$

(6) (20 pts) Given the nodes $x_0 < x_1 < \cdots < x_{12}$, we interpolate $f(x) = x^{13}$ with a polynomial $p(x)$ of degree 12 using these 13 nodes in $[-1, 1]$.
(a) What is a good upper bound for $|f(x) - p(x)|$ on $[-1, 1]$?

Solution: The error formula for polynomial interpolation gives

$$|f(x) - p(x)| = |\frac{f^{(13)}(\eta)}{13!}\prod_{i=0}^{12}(x - x_i)| = \prod_{i=0}^{12}|x - x_i|$$

and on the interval on the interval $[-1, 1]$ the best we can say is $|f(x) - p(x)| \leq 2^{13}$.

(b) What is the best possible bound for $\max_{-1\leq x\leq 1}|f(x) - p(x)|$ and for what set of nodes you have this optimal bound?

Solution: The optimal bound can be achieved only for Chebyshev nodes, i.e., when the nodes $\{x_i\}$ are the zeros of the Chebyshev polynomial $T_n(x) = \cos(n\cos^{-1}(x))$ and the bound for $n = 12$ is:

$$\max_{-1\leq x\leq 1}|f(x) - p(x)| = \max_{-1\leq x\leq 1}\prod_{i=0}^{12}|x - x_i| = \frac{1}{2^{12}}\max_{-1\leq x\leq 1}|T_{12}(x)| = \frac{1}{4096}.$$