p477/ #7

(A) $f'''(x) \approx \frac{1}{h^3}\left(f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x) \right)$

Taylor:

$$\begin{array}{r|l}
3* & f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \frac{h^4}{24} f^{(4)}(x) + \dots \\
-3* & f(x+2h) = f(x) + 2h f'(x) + \frac{4h^2}{2} f''(x) + \frac{8h^3}{6} f'''(x) + \frac{16h^4}{24} f^{(4)}(x) + \dots \\
1* & f(x+3h) = f(x) + 3h f'(x) + \frac{9h^2}{2} f''(x) + \frac{27h^3}{6} f'''(x) + \frac{81h^4}{24} f^{(4)}(x) + \dots
\end{array}$$

$$f(x+3h) - 3f(x+2h) + 3f(x+h) = f(x) + 0 \cdot f'(x) + 0 \cdot f''(x) + h^3 \underbrace{\left(\frac{1}{2} - \frac{8}{2} + \frac{27}{6} \right)}_{=1} f'''(x)$$

$81 - 45 = 36$

$$+ \frac{h^4 f^{(4)}(x)}{24} \left(\underbrace{3 - 3 \cdot 16 + 81}_{\neq 0} \right) + O(h^5)$$

Then the error for formula (A) is

$$\underline{\text{Error}(\#A) = \frac{9h}{8} f^{(4)}(X) + O(h^2)} = O(h)$$

(B) $f'''(x) \approx \frac{1}{2h^3}\left[f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h) \right]$

Taylor:

$$\begin{array}{r|l}
(\mp 2)x & f(x \pm h) = f(x) \pm h f'(x) \pm \frac{h^2}{2} f''(x) \pm \frac{h^3}{6} f'''(x) \pm \frac{h^4}{24} f^{(4)}(x) \pm \frac{h^5}{120} \dots \\
(\pm 1)* & f(x \pm 2h) = f(x) \pm 2h f'(x) \pm \frac{4h^2}{2} f''(x) \pm \frac{8h^3}{6} f'''(x) \pm \frac{16h^4}{24} f^{(4)}(x) \pm \frac{32h^5}{120} f^{(5)}(x)
\end{array}$$

$$f(x+2h) - f(x-2h) - 2(f(x+h) - f(x-h)) = \cancel{4h f'(x)} + \frac{8h^3}{3} f'''(x) + \frac{32h^5}{60} f^{(5)}(x) + O(h^7)$$

$$-2\left(\cancel{2h f'(x)} + \frac{h^3}{3} f'''(x) + \frac{h^5}{60} f^{(5)}(x) + O(h^7) \right)$$

Then $f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h) = 2h^3 f'''(x) + O(h^5)$

$\text{Error}(B) = O(h^2) \longrightarrow$ <u>better accuracy!</u>

p.477/#14 → Taylor series ...

$$f'(x) - \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h} = O(h^2)$$

$$= \frac{h^2}{3} f^{(3)}(\xi)$$

p.477/#16 Taylor ....

$$f''(x) - \frac{f(x) - 2f(x+h) + f(x+2h)}{h^2} = O(h)$$ Not very accurate!

#18 – Taylor (similar to 14)

Error $= \frac{h^2}{3} f'''(\xi)$ ξ is between x_n, x_{n-1}, x_{n-2}

So, if $|f'''(x)| \le C$ then

Error $= O(h^2) \longrightarrow 0$

p. 488 / #2 — we did in class:

$$\int_{-1}^{1} x^k \, dx = \frac{1}{3} f(-1) + \frac{4}{3} f(0) + \frac{1}{3} f(1)$$

$\underline{k = 0, 1, 2, 3}$ $\Rightarrow$ basis for all cubic functions.

#4 $$\int_0^1 f(x)\,dx = \frac{1}{90}\left[7 f(0) + 32 f\left(\tfrac{1}{4}\right) + 12 f\left(\tfrac{1}{2}\right) + 32 f\left(\tfrac{3}{4}\right) + 7 f(1)\right]$$

Verify for $f(x) = 1$, $\boxed{x - \frac{1}{2}}$, $\left(x - \frac{1}{2}\right)^2$, $\boxed{x(x-1)\left(x - \frac{1}{2}\right)}$,

and $\left(x - \frac{1}{2}\right)^4$.

a basis for Π_4.

$\int f(x) = 1$; 0 , $\frac{1}{12}$, 0 , $\frac{1}{32 \cdot 5}$ .....

#10 $$f(x) \approx f\left(\tfrac{1}{3}\right) \frac{x - \frac{2}{3}}{\frac{1}{3} - \frac{2}{3}} + f\left(\tfrac{2}{3}\right) \frac{x - \frac{1}{3}}{\frac{2}{3} - \frac{1}{3}}$$

$$\int_0^1 f(x) \approx f\left(\tfrac{1}{3}\right) \underbrace{\int_0^1 \frac{x - \frac{2}{3}}{-\frac{1}{3}}\,dx}_{\text{compute}} + f\left(\tfrac{2}{3}\right) \underbrace{\int_0^1 \frac{x - \frac{1}{3}}{\frac{1}{3}}\,dx}_{\text{compute}}$$

#13 $$\int_0^2 x f(x)\,dx = \overset{=0}{A} f(0) + \overset{=4/3}{B} f(1) + \overset{=2/3}{C} f(2)$$ — either by Lagrange or system.

Take $f = 1$; $f = x$ | $f(x) = (x-1)(x-2)$

$\int_0^2 x\,dx = 2 = \underline{A + B + C}$ | $\frac{8}{3} = B + 2C$ | $\underline{\int x f(x) = 0} = A \cdot 2 + 0$

$\underline{B + C = 2}$ $C + 2 = \frac{8}{3}$ $A = 0$

$C = \frac{2}{3}$ $B = \frac{4}{3}$ $\boxed{\text{DAC} = 2 \text{ only}}$

#13 continued

$$\int_0^2 x \cdot \underbrace{(x-1)^2(x-2)}_{f(x)} \, dx < 0 = \underline{A f(0) + B f(1) + C f(2)}$$

Not exact for Π_3, exact for $\Pi_2 \Rightarrow DAC = 2$

22 $\int_0^1 f(x) \approx \alpha f(x_0) + \alpha f(x_1)$

$\int 1 \, dx = 1 = \alpha + \alpha \Rightarrow \alpha = \frac{1}{2}$

$\int (x - \frac{1}{2}) \, dx = 0 = \alpha (x_0 - \frac{1}{2}) + \alpha (x_1 - \frac{1}{2})$

$0 = \frac{x_0 + x_1}{2} - \frac{1}{2} \Rightarrow \underline{x_0 + x_1 = 1}$

$\int_0^1 (x - \frac{1}{2})^2 \, dx = \frac{(x-\frac{1}{2})^3}{3} \Big|_0^1 = 2 \cdot \frac{1}{8} \cdot \frac{1}{3} = \frac{1}{12} = \frac{1}{2}\left((x_0 - \frac{1}{2})^2 + (x_1 - \frac{1}{2})^2\right)$

Solve ... $\Rightarrow$ $\underline{x_0 = \frac{1}{2} \pm \frac{\sqrt{3}}{6}}$, $\underline{x_1 = 1 - x_0}$

! <u>This is the Gaussian rule with two points!</u>

$\underline{A f(x_0) + B f(x_1)} \rightarrow \underline{A = B!}$ $n=1$ $\underline{2n+1 = 3}$

$\underline{DAC = 3}$ exact for Π_3

p499.

$\boxed{\# 11}$ $\int_0^2 f(x) \approx f(\alpha) + f(2-\alpha)$

exact for $\Pi_3 \iff$ Gaussian rule $\underline{A f(x_0) + B f(x_1)}$

$A = B = 1$ & $\underline{x_{0,1} = 1 \pm \frac{\sqrt{3}}{3}}$

$\boxed{\# 14.}$ Take $p(x) = (x-x_0)\cdots(x-x_n)$ and decompose:

$\underline{f(x) = q(x)\,p(x) + r(x)}$ If $f \in \Pi_{2n+1}$ we have $\underline{q, r \in \Pi_n}$ and

$\underline{f(x_i) = r(x_i)}$. Then (just like in class)

$\int_a^b w(x) f(x)\,dx = \sum_{i=0}^n A_i f(x_i)$ (given for $\forall f \in \underline{\Pi_{n+1}}$)

$= \int_a^b w(x)\,q(x)\,p(x)\,dx + \int_a^b w(x)\,r(x)\,dx = \sum A_i\, r(x_i)$

$+ \int_a^b q(x)\,p(x)\,w(x)\,dx$

Now $\sum_{i=0}^n A_i\, r(x_i) = \sum_{i=0}^n A_i f(x_i)$ because $r(x_i) = f(x_i)$

Then $\int_a^b w(x)\, q(x)\, p(x)\,dx = 0$

$\underline{\forall q \in \Pi_n}$ (because f can be any polynomial in Π_{2n+1} then q can be arbitrary polynomial in Π_n)

p. 500/ #15 → Simpson's rule again

$$\left\{ \frac{5}{9}, \frac{8}{9}, \frac{5}{9} \right\} = \text{coeff.}$$

(#17) Gaussian rule $n=4$ $(x_0, \dots, x_4)$ = zeros of $p_4(x)$

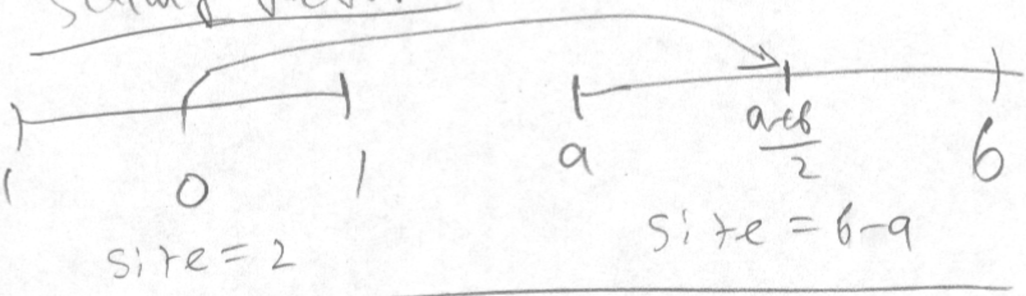
$\{p_n(x)\}_{n=0}^{\infty}$ = orthogonal polynomials with the weight $w(x) = x$ on $[a,b] = [0,1]$

i.e. $\langle p_i, p_j \rangle_w = 0 \quad i \neq j$ $\quad \langle f, g \rangle = \int_0^1 x\, f(x) g(x)\, dx$

(#22) $[-1, 1] \longrightarrow [a, b]$ → I did it in class

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2} \left[\frac{5}{9} f\left(\frac{a+b}{2} - \frac{b-a}{2}\sqrt{\frac{3}{5}} \right) + \frac{8}{9} f\left(\frac{b+a}{2} \right) + \frac{5}{9} f\left(\frac{a+b}{2} + \frac{b-a}{2}\sqrt{\frac{5}{9}} \right) \right]$$

scaling factor



(28) Take $p(x) = (x-x_1)^2 (x-x_2)^2 \cdots (x-x_n)^2$

nodes $= x_1, \dots, x_n$ (n nodes)

Then $\int p(x)\,dx > 0 = \sum_{j=1}^{n} A_j\, p(x_j) \Rightarrow$ NOT exact