(1) = §6.2 #23

-1	2	-1	1	-2	2
0	1	1	-5	6	
1	2	-9	13		
2	-7	17			
3	10				

$p(x) = 2 - (x+1) + x(x+1) - 2x(x+1)(x-1)$ | check

$2 = f(-1)$; $f[-1,0] = -1$; $f[-1,0,1] = 1$; $f[-1,0,1,2] = -2$

$f[-1,0,1,2,3] = 2$

$\Rightarrow q(x) = p(x) + 2 \cdot x(x+1)(x-1)(x-2)$

Check: $q(3) = 10 \stackrel{?}{=} 2 - 4 + 12 - 2\cdot3\cdot4\cdot2 + 2\cdot3\cdot4\cdot2\cdot1$

$10 \stackrel{?}{=} 10 - \cancel{48} + \cancel{48}$ ✓

Or just say $q(x)$ must be $= p(x) + A\cdot(x-1)x(x+1)(x-2)$

and $q(3) = p(3) + 24A$ ..

So $A = \dfrac{-p(3)+10}{24} = 2$

(2)

x	f	$f[\cdot,\cdot]$				
0	0	0	1/2	1/2	-5/2	6
0	0	0	1	-2	7/2	
0	0	1	-1	3/2		
1	1	0	1/2			
1	1	0				
1	1					

$P_5(x) = 0 + 0\cdot(x-0) + \frac{1}{2}(x-0)^2 + \frac{1}{2}(x-0)^3 - \frac{5}{2}(x-0)^3(x-1) + 6(x-0)^3(x-1)^2$

So, $P_5(x) = \dfrac{x^2}{2} + \dfrac{x^3}{2} - \dfrac{5x^3(x-1)}{2} + 6x^3(x-1)^2$

Check: $P(0) = 0$ ✓; $P'(0) = 0$ ✓; $P''(0) = 1$ ✓

$P(1) = \frac{1}{2} + \frac{1}{2} = 1$ ✓; $P'(1) = 1 + \frac{3}{2} - \frac{5\cdot1}{2} + \cancel{6\cdot0} = 0$ ✓

$P''(1) = 1 + 3 - \frac{5}{2}(\underbrace{4\cdot3\cdot1 - 3\cdot2}_{12-6}) + 6\cdot2\cdot1\cdot1$

$= 1 + 3 - 15 + 12 = 1$ ✓

Take $P_5(x) = Cx^2 + D\cdot x^3 + E x^3(x-1) + F\cdot x^3(x-1)^2 + A + Bx$

$P(0) = 0$ $P'(0) = 0$

P_5 is uniquely determined by the coefficients ...

Take the functions: $1, x, x^2, x^3, x^3(x-1), x^3(x-1)^2$

Span $= \Pi_5$ $\rightarrow$ need to show that A, B, C, D, E, F are unique.

Linear Algebra: $p_5(x) \equiv 0 \iff A = B = C = D = E = F = 0$

Given zero initial data for $\left\{ \begin{array}{l} p, p', p'' \text{ at } x=0 \\ p, p', p'' \text{ at } x=1 \end{array} \right\}$ we have the following system

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 & 1 \\ 0 & 0 & 2 & 6 & 6 & 2 \end{pmatrix}}_{M} \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} p(0) \\ p'(0) \\ p''(0) \\ p(1) \\ p'(1) \\ p''(1) \end{pmatrix} \iff \text{unique solution iff } \det M \neq 0$$

$$\det M = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \det\begin{pmatrix} 2 & 6 \\ 1 & 1 \end{pmatrix} \det\begin{pmatrix} 1 & 1 \\ 6 & 2 \end{pmatrix} = 1(-4)\cdot(-4) = \underline{16 \neq 0}$$

---

(3) $S(x) = \begin{cases} S_0(x) & \text{if } 0 \le x < \frac{1}{2} \\ S_1(x) & \text{if } \frac{1}{2} < x \le 1 \end{cases}$ use $S(0) = 0$, $S(1) = 0$

Take $\begin{cases} S_0(x) = ax + bx^2 + cx^3 \\ S_1(x) = d(x-1) + e(x-1)^2 + f(x-1)^3 \end{cases}$

The spline must satisfy:

$\boxed{S_0(\tfrac{1}{2}^-) = S_0(\tfrac{1}{2}^+) = 1}$

$\boxed{S_0'(\tfrac{1}{2}^-) = S_0'(\tfrac{1}{2}^+)}$

$\boxed{S_0''(\tfrac{1}{2}^-) = S_0''(\tfrac{1}{2}^+)}$

$\boxed{S_0(0) = S_0(1)}$ ✓

$\boxed{S_0'(0) = S_0'(1)}$

$\boxed{S_0''(0) = S_0''(1)}$

The solution is:

$\boxed{S_0(x) = 12x^2 - 16x^3}$

$\boxed{S_1(x) = 12(x-1)^2 - 16(x-1)^3}$ → see next page or just solve

$S_0'(x) = a + 2b x + 3c x^2$

$S_1'(x) = d + 2e(x-1) + 3f(x-1)^2$

$S_0''(x) = 2b + 6c x$

$S_1''(x) = 2e + 6f(x-1)$

$S_0'(0) = a = S_1'(1) = d$

$S_0''(0) = 2b = S_1''(1) = 2e$

So $a = d$ & $b = e$, i.e., $S(x) = \begin{cases} ax + bx^2 + cx^3 & \text{if } x < \frac{1}{2} \\ a(x-1) + b(x-1)^2 + f(x-1)^3 & \text{if } x > \frac{1}{2} \end{cases}$

$S(\frac{1}{2}) = 1 = \boxed{\frac{a}{2} + \frac{b}{4} + \frac{c}{8} = -\frac{a}{2} + \frac{b}{4} - \frac{f}{8}} \Rightarrow \boxed{a + \frac{c+f}{8} = 0}$

$S'(\frac{1}{2}) = \cancel{a} + 2b\cdot\frac{1}{2} + 3c\cdot\frac{1}{4} = \cancel{a} + 2b(-\frac{1}{2}) + 3f(-\frac{1}{2})^2$

$S''(\frac{1}{2}) = \cancel{2b} + 6c\cdot\frac{1}{2} = \cancel{2b} + 6f(-\frac{1}{2}) \Rightarrow$ $c + f = 0$, $a = 0$

$2b + \frac{3c}{2} = 0$ $\quad \boxed{b = -\frac{3c}{4}}$ $\quad b = 12$

$1 = \frac{2b + c}{8}$ $\quad 2b + c = 8$ $\quad -\frac{3c}{2} + c = 8$ $\quad c = -16$

Check $\quad S(0) = 0 = S(1)$
$S(\frac{1}{2}) = 3 - 2 = 1$

$S(x) = \begin{cases} 12x^2 - 16x^3 \\ 12(x-1)^2 + 16(x-1)^3 \end{cases}$

$S'(0) = 0 = S'(1)$

$S' = \begin{cases} 24x - 48x^2 \\ 24(x-1) + 48(x-1)^2 \end{cases}$

$S'(\frac{1}{2}) = 12 - 12 = 0$ ✓
$= -12 + 12 = 0$

$S'' = \begin{cases} 24 - 96x \\ 24 + 96(x-1) \end{cases}$

$S''(\frac{1}{2}) = 24 - 48 = -24$
$= 24 - 48 = -24$ ✓

Solution: $S(x) = \begin{cases} S_0(x) = 12x^2 - 48x^3 & \text{if } 0 \le x < \frac{1}{2} \\ S_1(x) = 12(x-1)^2 - 48(x-1)^3 & \text{if } \frac{1}{2} < x \le 1 \end{cases}$

(4)

$$2 * \quad f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + O(h^4)$$

$$\ominus \quad f(x+2h) = f(x) + 2h f'(x) + \frac{(2h)^2}{2} f''(x) + \frac{(2h)^3}{6} f'''(x) + O(h^4)$$

$$\Rightarrow \quad 2 f(x+h) - f(x+2h) = f(x) + (h^2 - 2h^2) f''(x) + \frac{2-8}{6} h^3 f'''(x) + O(h^4)$$

Then
$$f''(x) = \frac{f(x) - 2f(x+h) + f(x+2h)}{h^2} \underbrace{- \frac{h^3 f'''(x)}{h^2} + \frac{O(h^4)}{h^2}}_{= O(h)}$$

Or we can write that

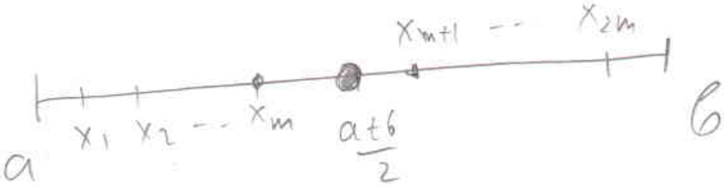
$$f''(x) - \frac{f(x) - 2f(x+h) + f(x+2h)}{h^2} = -h f'''(x) + O(h^2)$$
$$= O(h)$$

---

(5) <u>Two different proofs</u>

(I) $\int f(x) \approx \int P_n(x)$ where $P_n(x)$ interpolates $f(x)$

$n = 2m$ <u>interpolation points</u>



(*) $\boxed{\frac{x_{i+m} + x_i}{2} = \frac{a+b}{2}}$

$$\int_a^b \left(x - \frac{a+b}{2}\right)^{n+1} dx = 0$$ and <u>the num. rule = 0</u>

$$\int_a^b P_n(x) = \sum_{i=1}^{2m} f(x_i) \underbrace{\int_a^b \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} dx}_{= A_i}$$

$$(*) \Leftrightarrow A_i = A_{i+m} \Rightarrow \sum_{i=1}^{2m} f(x_i) = 0$$ if f = odd

II - proof Interpolate f by $p_n(x)$ and $q_{n+1}(x)$

p_n — interpolates at $x_1, x_2, \dots, x_{2m}$

q_{n+1} — interpolates at $x_1, x_2, \dots, x_{2m}$ and $\frac{a+b}{2}$!

Then $\int_a^b f(x)\,dx = \int_a^b q_{n+1}(x)$ for all $f(x) \in \Pi_{n+1}$

Recall that $q_{n+1}(x) = p_n(x) + A(x-x_1)\dots(x-x_{2m})$

$A = f[x_1, x_2 \dots x_{2m}]$

Put the interp. points in the order $x_1, x_2, \dots, x_{2m}, \frac{a+b}{2}$

and use the divided difference formula!

Now: $\int_a^b q_{n+1}(x) = \int_a^b p_n(x) + A \int_a^b \prod_{j=1}^{2m} (x-x_j)\,dx$

and $\int_a^b \underbrace{\prod_{j=1}^{2m} (x-x_j)}_{\text{odd function}}\,dx = 0$ from symmetry

Hence $\int_a^b f(x)\,dx \approx \int_a^b q_{n+1}(x) = \int_a^b p_n(x)$

and DAC $\geq n+1$