

August 30, 2011

L#1

## Systems of linear equations

$\mathbb{R}^n$  is the space of  $n$ -dimensional vectors (we shall use column notation)  $\mathbb{R}^n \ni x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$   $x_j \in \mathbb{R}$  with standard algebra:  $x+y = \{x_i+y_i\}_{i=1}^n$   $\alpha x = \{\alpha x_i\}$  etc.

$\mathbb{R}^{n \times m}$  is the space of  $n \times m$  rectangular tables

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} \in \mathbb{R}^{n \times m} \quad \begin{array}{l} n\text{-rows} \\ m\text{-columns} \end{array}$$

with similar algebra  $A = \{a_{ij}\}_{\substack{i=1, \dots, n \\ j=1, \dots, m}}$   $B = \{b_{ij}\}$

①  $A+B = \{a_{ij}+b_{ij}\}$   $\boxed{n=m}$  square matrices

②  $\lambda A = \{\lambda a_{ij}\}$

③  $AB = \left\{ \sum_{k=1}^n a_{ik} b_{kj} \right\}_{i,j=1}^n$

If  $n \neq m$  we can multiply  $n \times m$  by  $m \times s$  matrices

$A \in \mathbb{R}^{n \times n}$   $Ax = b$   $b \in \mathbb{R}^n$  given is a system of  $n$  equations  
inner product in  $\mathbb{R}^n \ni x, y$   $(x, y) = \sum x_i y_i$

Some simple matrices

I - identity matrix

$$a_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

D - diagonal matrix

$$a_{ij} = \begin{cases} \neq 0 & i=j \\ 0 & i \neq j \end{cases}$$

L - lower triangular matrix

$$a_{ij} = 0 \quad \text{if } i < j$$

U - upper triangular matrix

$$a_{ij} = 0 \quad \text{if } i > j$$

A<sup>T</sup> - transposed to matrix A

$$(A^T)_{ij} = \{a_{ji}\}$$

$$\text{System } Ax = b \quad \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\begin{matrix} (n \times n) & \times & (n \times 1) & = & (n \times 1) \\ A & & x & & b \end{matrix}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases}$$

Equivalent systems  $Ax = b$   $Bx = d$  equivalent

Elementary transformations: interchanging two equations  
adding two equations, ...

Elementary transformations produce equivalent systems

Easy to solve systems  $Dx = b$ ,  $Lx = b$ ,  $Ux = b$

Let  $A, B \in \mathbb{R}^{n \times n}$ . If  $AB = I \Rightarrow BA = I$   
we call  $B$ -inverse  
of  $A$

Th. If  $AB = I$  then  $BA = I$ , i.e. there is one inverse.

Proof.  $C = BA - I + B$      $AC = \underbrace{ABA} - A + AB = AB$

Given } form  
 $AB = I$

Then if we show that  $A$  may have at most one right inverse the result will follow.

$$C = B$$

$$BA = I \Rightarrow B \text{ is left inverse}$$

Definition: If  $AB = BA = I$  then we call  $B$  inverse of  $A$ , and denote it by  $A^{-1}$  and say  $A$  is nonsingular

Definition The determinant of  $A$  is a number defined by

$$\det A = \sum_{\text{overall possible permutations of } (1, \dots, n)} (-1)^{[i_1, \dots, i_n]} a_{1i_1} a_{2i_2} \dots a_{ni_n}$$

$[i_1, \dots, i_n]$  - number of permutations

Ex.  $n=2$      $\det A = (-1)^0 a_{11}a_{22} + (-1)^1 a_{12}a_{21}$   
 $= a_{11}a_{22} - a_{12}a_{21}$

$(1,2), (2,1)$   
these are all possible permut-  
of  $(1,2)$

### A constructive definition of $\det A$

$M_{ij}$  is the determinant of the  $(n-1) \times (n-1)$  matrix obtained from  $A$  by deleting the  $i$ -th row and  $j$ -th column

$$\begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{ii} & & a_{ij} & & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix} = M_{ij} \text{ - minor}$$

$$\det A = \sum_{j=1}^n (-1)^{i+j} \det M_{ij} \text{ for any } i=1, \dots, n$$

A formula for  $A^{-1}$  via minors

$$(A^{-1})_{ij} = \frac{(-1)^{i+j} \det M_{ji}}{\det A}$$

$$Ax=b \Rightarrow x=A^{-1}b$$

Easy to solve systems  $Lx=b$

$$\begin{aligned} l_{11}x_1 &= b_1 \\ l_{21}x_1 + l_{22}x_2 &= b_2 \\ l_{i1}x_1 + \dots + l_{ii}x_i &= b_i \\ l_{n1}x_1 + \dots + l_{nn}x_n &= b_n \end{aligned}$$

$$l_{ii} \neq 0$$

$$x_1 = b_1/l_{11} \quad x_2 = (b_2 - l_{21}x_1)/l_{22}$$

long operations

$$x_i = (b_i - \sum_{j=1}^{i-1} l_{ij}x_j)/l_{ii} \quad i=1, \dots, n$$

$$\sum_{i=1}^n i = \frac{n(n+1)}{2} = O\left(\frac{n^2}{2}\right)$$

Main Theorem: Let  $A \in \mathbb{R}^{n \times n}$ . The following are equivalent

- (1)  $A$  is nonsingular, i.e.  $A^{-1}$  exists
- (2)  $\det A \neq 0$
- (3) the rows of  $A$  form a basis for  $\mathbb{R}^n$
- (4) the columns of  $A$  form a basis for  $\mathbb{R}^n$
- (5) the equation  $Ax = 0$  has only one solution  $x = 0$
- (6) the equation  $Ax = b$  has unique solution for any  $b \in \mathbb{R}^n$ .

Definition:  $A \in \mathbb{R}^{n \times n}$  is called symmetric and positive definite if

- (1)  $A$  symmetric,  $A = A^T$
- (2)  $x^T A x > 0$  for any  $x \in \mathbb{R}^n$   $x \neq 0$

Theorem: If  $A$  is an SPD matrix then

- (1) all principle minors are SPD submatrices in particular  $a_{ii} > 0$  for  $i=1, \dots, n$
- (2)  $A^{-1}$  exists (i.e.  $A$  is nonsingular)
- (3)  $A^{-1}$  is an SPD matrix

Proof

(1)  $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$  -  $i$ -th position  $e_i^T A e_i = a_{ii} > 0$

(2) assume  $A^{-1}$  does not exist  $Ax_0 = 0$  will have a nontrivial solution  $x_0 \neq 0$   
take  $x_0^T A x_0 = 0$  contrary to assumption that  $> 0$   
therefore  $A^{-1}$  should exist

(3)  $A^{-1}$  is symmetric; is it PD?

$x^T A^{-1} x = y^T A y > 0$  OR

$A^{-1} x = y$   $x = A y$   $x^T = y^T A^T = y^T A$

Symmetric matrix  $a_{ij} = a_{ji} \quad i, j = 1, \dots, n$

$$(Ax, y) = \sum_i \sum_j a_{ij} x_j y_i = \sum_i \sum_j a_{ij} y_i x_j = (x, Ay)$$

$$\sum_j \sum_i a_{ji} y_i x_j = (Ay, x)$$

For general matrices we have

$(Ax, y) = (x, A^T y)$  show this

A - spd      A is a linear operator  $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$

A symmetric in  $\langle \cdot, \cdot \rangle$  iff  $\langle \cdot, \cdot \rangle$  inner product

$\langle Ax, y \rangle = \langle x, Ay \rangle \quad \forall x, y \in \mathbb{R}^n$

Theorem: Let  $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be an SPD matrix with respect to  $\langle \cdot, \cdot \rangle$  in  $\mathbb{R}^n$ . Then A has a complete set of eigenvectors that can be orthonormalized in  $\langle \cdot, \cdot \rangle$ -inner product

$A\psi_i = \lambda_i \psi_i \quad \langle \psi_i, \psi_j \rangle = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \quad \lambda_j > 0$

$\{\psi_j\}_{j=1}^n$  - form a basis in  $\mathbb{R}^n$ , i.e.

for any  $x \in \mathbb{R}^n$   $\exists!$  representation

$x = \sum c_j \psi_j \quad c_j = \langle x, \psi_j \rangle \quad j = 1, \dots, n$