

Basic iterative methods (from the past)

We are studying the iteration methods

$$x^{(k+1)} = x^{(k)} + \beta(f - Ax^{(k)}), \quad x^{(0)} \text{ given} \quad k=0, 1, \dots$$

Or in the form after defining $Q^{-1} = B$

$$Qx^{(k+1)} = (Q - A)x^{(k)} + f \quad x^{(0)} \text{ given}$$

Q is called splitting of A

B is called preconditioner of A

Various splittings lead to different methods. Recall that we want

(1) B to be easy to compute with

(2) Iteration to converge fast

Splittings of $A = D - L - U$

$$Q = \frac{1}{\tau} \quad \tau \text{-iteration parameter} \quad \begin{matrix} \text{diagonal lower} \\ \text{upper triangular} \end{matrix} \quad \text{Richardson method}$$

$$Q = D$$

Jacobi method

$$Q = D - L \quad (\text{or } Q = D - U)$$

Gauss-Seidel

$$Q = \frac{1}{\omega} (D - \omega L) \quad 0 < \omega < 2$$

SOR

$$Q = \frac{1}{\omega} (D - \omega U)$$

these are all stationary processes

The concept of nonstationary process of iterations is when Q depends on k , then

$$Q_k(x^{(k+1)} - x^{(k)}) = B - Ax^{(k)} \quad k=0, \dots$$

The simplest nonstationary process is Richardson type

$$Q_k = \frac{1}{\tau_k} I \quad \tau_k - \text{iteration parameter}$$

In general, the analysis of any iteration method is quite difficult, especially when one needs to choose in "an optimal" way a parameter (or parameters) and there are no complete theoretical results.

However, for some classes of matrices this could be done in a nice way. Such class of matrices are SPD matrices.

The simplest case are matrices symmetric in the standard way, namely, $a_{ij} = a_{ji}$ or in Euclidean inner product $(x, y) = \sum_{i=1}^n x_i y_i$, $x, y \in \mathbb{R}^n$: $(Ax, y) = (x, A_y)$.

In \mathbb{R}^n one can introduce various inner products. For example, if B is an SPD matrix, then

$$(Bx, y) \stackrel{\det}{\equiv} (x, y)_B \text{ defines an inner product in } \mathbb{R}^n.$$

Note that if A & B are SPD matrices AB is not an symmetric matrix (in general) in the Euclidean sense. However it is symmetric in the B -inner product

Indeed

$$(ABx, y)_B = (BABx, y) = (Bx, ABy) = (x, ABy)_B$$

Definition: Let $\langle \cdot, \cdot \rangle$ be an inner product in \mathbb{R}^n . Then A is symmetric in $\langle \cdot, \cdot \rangle$ if $[Ax, y] = [x, Ay]$ for any $x, y \in \mathbb{R}^n$. It is called positive definite if $[Ax, x] > 0$, $\forall x \in \mathbb{R}^n$.

Theorem: Let $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be SPD with respect to the inner product $\langle \cdot, \cdot \rangle$ on \mathbb{R}^n . Then A has complete set of eigenvectors that can be orthonormalized, i.e.

$$A\psi^{(j)} = \lambda_j \psi^{(j)} \quad \langle \psi^{(j)}, \psi^{(i)} \rangle = \delta_{ij} \quad \lambda_j > 0$$

Now consider the iteration method for A an SPD

$$x^{(k+1)} = (I - BA)x^{(k)} + Bb \quad \text{where } B \text{ is an SPD}$$

We know that for the error $e^{(k)} = x^{(k)} - x$ we have

$$e^{(k+1)} = (I - BA)e^{(k)} \quad k=0, 1, \dots \quad e^{(0)} \text{ fixed}$$

Let $\langle x, y \rangle = (Ax, y)$ be an inner product in \mathbb{R}^n so that BA is symmetric with respect to $\langle \cdot, \cdot \rangle$.

④

Denote by (λ_j, ψ_j) the eigenpairs

$$BA\psi_j = \lambda_j \psi_j \quad j=1, \dots, n \quad \langle \psi_i, \psi_j \rangle = \delta_{ij}.$$

$\{\psi_j\}$ form a basis for \mathbb{R}^n .

Then

$$e^{(k)} = \sum_{j=1}^n c_j^{(k)} \psi_j$$

$$\begin{aligned} e^{(k+1)} &= \sum_j (I - BA) c_j^{(k)} \psi_j = \sum_j c_j^{(k)} (1 - \lambda_j) \psi_j \\ &= \sum_j g_j^{(k+1)} \psi_j \end{aligned}$$

Thus

$$g_j^{(k+1)} = (1 - \lambda_j) c_j^{(k)}$$

$$\begin{aligned} \langle e^{(k)}, e^{(k)} \rangle &= (Ae^{(k)}, e^{(k)}) = \|e^{(k)}\|_A^2 = \left(\sum_j c_j^{(k)} A\psi_j, \sum_j c_j^{(k)} \psi_j \right) \\ &= \sum_j c_j^{(k)} (A\psi_j, \psi_j) = \sum_{j=1}^n c_j^{(k)} \end{aligned}$$

$$\|e^{(k+1)}\|_A^2 = \sum_j (1 - \lambda_j)^2 c_j^{(k)} \leq \max_j |1 - \lambda_j|^2 \|e^{(k)}\|_A^2$$

(*)

$$\|e^{(k+1)}\|_A \leq \max_j |1 - \lambda_j| \|e^{(k)}\|_A$$

Denote by $\delta = \max_j |1 - \lambda_j|$. Since

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$$

obviously

$$\delta = \max(1 - \lambda_1, 1 - \lambda_n)$$

Therefore the goal is to choose B in such way so $\delta < 1$ and as small as possible.

Now let us consider some examples:

Richardson iteration for A an SPD matrix

$$Q = \frac{1}{\tau} I \quad B = \tau I \quad \text{so that}$$

$$x^{(k+1)} = (I - \tau A)x^{(k)} + \tau b \quad \begin{matrix} \tau - \text{scaling} \\ \text{parameter} \end{matrix}$$

$BA = \tau A \Rightarrow \{\tau \lambda_j\}$ are the eigenvalues of BA

Then (*) produces

$$\|e^{(k+1)}\|_A \leq \max_j |1 - \tau \lambda_j| \|e^{(k)}\|_A$$

Similarly one gets

$$\|e^{(k+1)}\|_2 \leq \max_j |1 - \tau \lambda_j| \|e^{(k)}\|_2$$

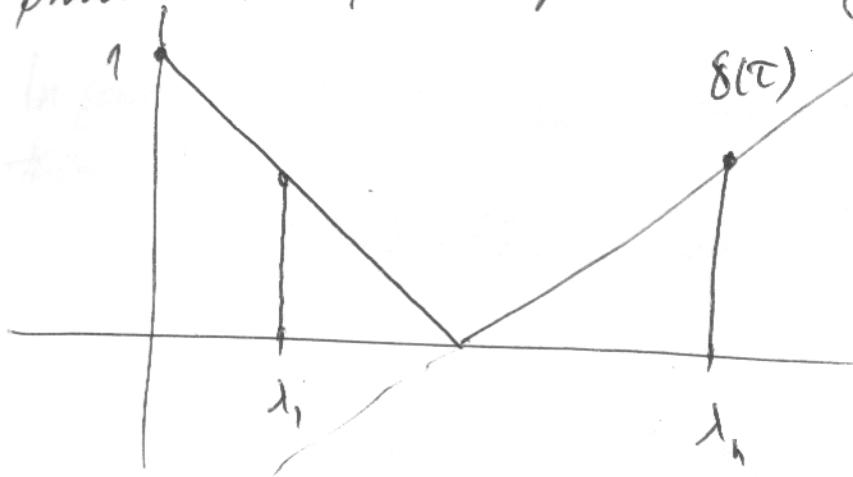
(6)

What values of τ will produce a convergent iterative method? Obviously if $\tau = \frac{1}{2} \frac{1}{\lambda_n}$ then

$$\delta = \max \{ |1-\tau\lambda_1|, 1, |1-\tau\lambda_n| \} = \left| 1 - 2 \frac{\lambda_1}{\lambda_n} \right|$$

Obviously for $\lambda_n \gg \lambda_1$, $\delta \approx 1$ and the convergence will be slow.

What would be the best value of τ , that will produce the fastest possible convergence?



the best $\delta(\tau)$ will be when

$$\delta_{\text{opt}} = \frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1} = \frac{1 - \frac{\lambda_1}{\lambda_n}}{1 + \frac{\lambda_1}{\lambda_n}} \approx 1 - 2 \xi + O(\xi)^2$$

$$\frac{\lambda_1}{\lambda_n} = \xi \ll 1$$

$$\tau = \frac{2}{\lambda_1 + \lambda_n}$$

$$\delta_{\text{opt}} \approx 1 - \frac{\pi^2}{2(n+1)^2}$$

Example:

$$\Delta = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\lambda_j = \sqrt{4 - \sin^2 \frac{\pi j}{2(n+1)}} \quad j=1, \dots, n$$

$$\frac{\lambda_1}{\lambda_n} = \tan^2 \frac{\pi}{2(n+1)} \approx \frac{\pi^2}{4(n+1)^2} \quad \xi = \frac{(n+1)^2}{4}$$

(7)

Now consider the general iterative method in the form

$$x^{(k+1)} = G x^{(k)} + c$$

Definition: Spectral radius of G is $\rho(G)$

$$\rho(G) = \max \{ |\lambda| : \det |G - \lambda I| = 0 \} = \sigma(G)$$

In general $\rho(G)$ is not a norm. But if G is symmetric then

$$\rho(G) = \|G\|_2 = \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{\|Gx\|_2}{\|x\|_2}$$

Proof $Gx = \lambda x$ (λ, ψ_j) eigenpairs

$$\text{let } \lambda = \max_{\lambda \in \sigma(G)} |\lambda|$$

$$\text{any } x = \sum_{j=1}^n c_j \psi_j \quad \|x\|_2 = (\sum c_j^2)^{1/2}$$

$$Gx = G \sum_{j=1}^n c_j \psi_j = \sum c_j \lambda_j \psi_j$$

$$\|Gx\|_2^2 = \sum_j c_j^2 \lambda_j^2$$

$$\frac{\|Gx\|_2}{\|x\|_2} = \left(\frac{\sum c_j^2 \lambda_j^2}{\sum c_j^2} \right)^{1/2} \leq \lambda \quad x = \psi_n$$

$$\frac{\|G\psi_n\|}{\|\psi_n\|} = \lambda$$

Main facts about spectral radius $\rho(A)$.

Fact 1: $\rho(A)$ is in general not a norm (trivial)

Fact 2: If A is symmetric, i.e. $A = A^T$ then (trivial)

$$\rho(A) = \|A\|_2 \quad \text{${N \times N}$-Euclidean norm}$$

Fact 3: $\rho(A) = \inf \|A\|$

where \inf is taken over all possible subordinate matrix norms

Fact 4: Consider the iteration $x^{(k+1)} = Gx^{(k)} + c$. Then

$$\rho(G) < 1$$

is sufficient and necessary condition for the convergence of the iteration method

Fact 3': Fact 3 is equivalent $\rho(A) = \inf_{\|\cdot\|} \|A\|$

(1) $\rho(A) \leq \|A\|$ for any matrix norm

(2) $\forall \delta > 0 \exists \|\cdot\|_\delta$ matrix norm s.t.

$$\|A\|_\delta \leq \rho(A) + \delta \quad \forall$$