CATEGORICAL DIMENSION AS A QUANTUM STATISTIC AND APPLICATIONS

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Abstract. We discuss several useful interpretations of the categorical dimension of objects in a braided fusion category, as well as some conjectures demonstrating the value of quantum dimension as a quantum statistic for detecting certain behaviors of anyons in topological phases of matter. From this discussion we find that objects in braided fusion categories with integral *squared* dimension have distinctive properties. A large and interesting class of non-integral modular categories such that every simple object has integral squared-dimensions are the metaplectic categories, which we describe and complete their classification. We prove that any modular category of dimension $2^k m$ with m square-free and $k \leq 4$, satisfying some additional assumptions, is a metaplectic category. This illustrates anew that dimension can, in some circumstances, determine a surprising amount of the category's structure.

1. Introduction

Dimensions of simple objects in (spherical) fusion categories are one of the most ubiquitous invariants we encounter. Algebraically, a (quantum) dimension function on a fusion category $\mathcal{C}$ is a generalization of a linear character for a finite group: it is an assignment of a complex number to each object $X \in \mathcal{C}$ that obeys the fusion rules. As the character table of a finite group G contains a significant amount of information about G itself (e.g. whether G is abelian, simple, or perfect), it is natural to ask: What information is contained in the dimensions of simple objects in a category? That is, how much of the structure, and which properties are determined by dimensions? The two dimension functions that one is most often interested in are the categorical and Frobenius-Perron (FP) dimensions. Fortunately, in the physically-relevant unitary setting the FP and categorical dimension coincide. For our purposes it is the FP dimension that is most useful, so we will focus on this particular function. In what follows we will simply refer to the FP dimension as the dimension.

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Non-degenerate (unitary) ribbon fusion (i.e. modular) categories model certain topological phases of matter [32], where the simple objects label anyons (quasi-particles), and the isomorphism classes correspond to the anyon types, or colors. In this interpretation the quantum dimensions of simple objects correspond asymptotically to the dimensions of state spaces of n identical anyons in a disk with boundary labeled by the trivial object (vacuum anyon). One may also ask: What properties of an anyon are determined by their (label's) dimension? As we view anyon systems through the lens of category theory, this leads to mathematical questions and conjectures. There are three important and seemingly unrelated properties of an anyon that are (at least conjecturally) controlled by the dimension: abelianness, localizability and universality. This motivates our perspective that dimension is the central quantum statistic for anyonic systems.

A coarser invariant of a fusion category is the global (FP) dimension: the sum of the squares of the dimension of the simple objects. Still, some properties of a modular category are determined by the global dimension. It is well-known (see [7, Cor. 3.14]) that there are finitely many fusion categories with a given fixed global dimension. In some cases one can go a bit further to give a complete classification of modular categories of a given global dimension in terms of other well-studied categories. We illustrate this principle by classifying certain modular categories of dimension $16m$ with m square-free in terms of metaplectic modular categories.

2. Preliminaries

We assume basic knowledge of braided fusion categories, referring the reader to [2, 16] for notation and elementary notions. We have left a few verifications as exercises for readers new to the subject.

Let $\mathcal{C}$ be a fusion category of rank r with $X_0 = \mathbf{1}, X_1, \ldots, X_{r-1}$ a collection of representatives of the distinct isomorphism classes of simple objects. The Grothendieck semi-ring $K_0(\mathcal{C})$ of $\mathcal{C}$ encodes the fusion rules of $\mathcal{C}$: it is the based $\mathbb{Z}_+$-ring with basis the simple isomorphism classes of objects and the operations induced by the direct sum $\oplus$ and the tensor product $\otimes$ [33]. A *dimension function* on a fusion category $\mathcal{C}$ is a unital ring homomorphism $K_0(\mathcal{C}) \to \mathbb{C}$. Thus the collection of dimension functions on a fusion category only depends on $K_0(\mathcal{C})$. A fusion category with a *commutative* Grothendieck ring (e.g. a braided fusion category) has exactly as many dimension functions as it has simple isomorphism classes of objects (see e.g. [8, Theorem 2.3]). A non-commutative fusion category may admit only the trivial dimension function: Vec_G for a perfect group G is such an example, since a dimension function for Vec_G is a linear character of G.

We will describe three dimension functions each of a rather distinct nature, and then argue that they are identical in the physically relevant case.

FP-dimension The fusion rules $N_{i,j}^k := \dim \mathrm{Hom}(X_i \otimes X_j, X_k)$ supply us with an explicit realization of the left-regular representation of $K_0(\mathcal{C})$ via $X_i \to N_i$ where $(N_i)_{k,j} := N_{i,j}^k$.

Indeed, $X_i \otimes X_j \cong \bigoplus_k N_{i,j}^k X_k$ so that it suffices to check the corresponding matrix equation (exercise) and then extend to $K_0(\mathcal{C})$ linearly $X \to N_X$. The first dimension function is the *Frobenius-Perron (FP) dimension*, defined as $\mathrm{FPdim}(X) := \max \mathrm{Spec}(N_X)$ for simple, i.e. the maximal eigenvalue of the fusion matrix N_i for simple X. The existence of such an eigenvalue follows from the Perron-Frobenius theorem applied to the positive matrix $\sum_{i,j} N_i N_j N_i^T$ that commutes with each N_k. (exercise, see [16, Section 8]). The fact that FPdim is a dimension function and is uniquely determined by $\mathrm{FPdim}(\mathbf{1}) = 1$ and $\mathrm{FPdim}(X_i) > 0$ is found in [16].

Asymptotic Dimension In the categorical model for topological phases of matter, the state space of n anyons of type X on a disk with boundary labeled by the vacuum aynon $\mathbf{1}$ is $\mathrm{Hom}(\mathbf{1}, X^{\otimes n})$. How does $\dim_{\mathbb{C}} \mathrm{Hom}(\mathbf{1}, X^{\otimes n})$ grow with n? If X were simply a d_X-dimensional vector space over $\mathbb{C}$ and $\mathbf{1} = \mathbb{C}$ then $\dim_{\mathbb{C}} \mathrm{Hom}(\mathbf{1}, X^{\otimes n}) = d_X^n$ for all n. More generally a natural measure of the dimension of an object X in a fusion category is a constant $d_X > 0$ such that $\dim_{\mathbb{C}} \mathrm{Hom}(\mathbf{1}, X^{\otimes n})$ grows like d_X^n, i.e. the asymptotic dimension of $\mathrm{Hom}(\mathbf{1}, X^{\otimes n})$. Of course it can happen that $\dim_{\mathbb{C}} \mathrm{Hom}(\mathbf{1}, X^{\otimes n})$ is 0 for many n, but there is a minimal $k > 0$ such that $\dim_{\mathbb{C}} \mathrm{Hom}(\mathbf{1}, X^{\otimes kn}) > 0$ for all $n > 0$ (see [12, Lemma F.6]), so that we may instead define $d_X > 0$ to be a constant such that $\dim_{\mathbb{C}} \mathrm{Hom}(\mathbf{1}, X^{\otimes k(n+1)}) / \dim_{\mathbb{C}} \mathrm{Hom}(\mathbf{1}, X^{\otimes kn}) \approx (d_X)^k$. The Perron-Frobenius theorem applied to the fusion matrix N_X implies that $d_X = \mathrm{FPdim}(X)$ (exercise, compute $\dim_{\mathbb{C}} \mathrm{Hom}(\mathbf{1}, X^{\otimes kn})$ in terms of the entries of N_X). Thus the asymptotic dimension coincides with the FP-dimension and in particular is a dimension function.

We illustrate this with an example:

Example 2.1. *Consider $\mathcal{C} = \mathrm{Fib}$ the Fibonacci modular category. There are 2 colors $\mathbf{1}, X$ in $\mathcal{C}$. Since $X \otimes X = \mathbf{1} \oplus X$ then the fusion matrix is $N_X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and it has eigenvalues $\frac{1 \pm \sqrt{5}}{2}$. Therefore, $\dim(X) = \frac{1+\sqrt{5}}{2}$.*

The fusion rules of the category $\mathcal{C}$ are given by the Fibonacci numbers in the following way $X^{\otimes i} = F(i-1)\mathbf{1} \oplus F(i)X$. Then we can compute:

$$\lim_{i \to \infty} \frac{\dim \mathrm{Hom}_{\mathcal{C}}(X^{\otimes 2i}, \mathbf{1})}{\dim \mathrm{Hom}_{\mathcal{C}}(X^{\otimes 2(i-1)}, \mathbf{1})} = \lim_{i \to \infty} \frac{F(2i-2)}{F(2i-4)} = \left(\frac{1+\sqrt{5}}{2} \right)^2.$$

Categorical Dimension Now let $\mathcal{C}$ be a ribbon fusion category, and $X \in \mathcal{C}$. Denote the rigidity maps $coev_X : \mathbf{1} \to X \otimes X^*$ and $ev_X : X^* \otimes X \to \mathbf{1}$, braiding morphism $c_{X,X^*} : X \otimes X^* \cong X^* \otimes X$ and ribbon twist θ_X. Then, for any $f \in \mathrm{End}(X)$ we have a morphism

$$\mathrm{Tr}_{\mathcal{C}}(f) = ev_X c_{X,X^*} (\theta_X f \otimes \mathrm{Id}_{X^*}) coev_X \in \mathrm{End}(\mathbf{1}).$$

Now since $\mathrm{End}(\mathbf{1}) \cong \mathbb{C}$, with basis $\mathrm{Id}_{\mathbf{1}}$, we may define $\dim(X)$ to be the coefficient of $\mathrm{Id}_{\mathbf{1}}$ in $\mathrm{Tr}_{\mathcal{C}}(\mathrm{Id}_X)$. In [41] it is shown that *dim* is a dimension function (exercise: show that $\mathrm{Tr}_{\mathcal{C}}$ is multiplicative with respect to $\otimes$, using naturality of c and θ.) In fact, one may define a categorical dimension for any spherical fusion category using the pivotal trace, but we will focus on the braided case.

A Hermitian ribbon fusion category $\mathcal{C}$ is equipped with an additive involutive operation $\dagger$ on morphisms, $\dagger : \mathrm{Hom}(X,Y) \to \mathrm{Hom}(Y,X)$, that is compatible with $\otimes$, composition, braiding ($c_{X,Y}^{\dagger} = c_{X,Y}^{-1}$), twists and rigidity morphisms [40]. In particular $(f,g) := \mathrm{Tr}_{\mathcal{C}}(fg^{\dagger})$ is a non-degenerate Hermitian form on $\mathrm{Hom}(X,Y)$. If, in addition, $\dagger$ acts on $\mathbb{C}$ by complex conjugation and (f,g) is a positive definite form then we say $\mathcal{C}$ is unitary. In particular $(Id_X, Id_X) = \mathrm{Tr}_{\mathcal{C}}(Id_X) = \dim(X) > 0$ for all X, and hence for a unitary category $\dim(X) = \mathrm{FPdim}(X)$ for all X.

Unless otherwise stated, in the rest of this article we will assume that $\dim(X) = \mathrm{FPdim}(X)$ for all objects. In many situations (for example if $\mathrm{FPdim}(X)^2 \in \mathbb{Z}$ for all simple X) it is possible to replace the twists (or equivalently the spherical structure) on a ribbon fusion category by another (unique) choice to ensure that $\dim(X) = \mathrm{FPdim}(X)$ (see [16]). On the other hand, there are examples [36] of modular categories whose underlying fusion rules do not admit a unitary ribbon categorification. The positivity of dim actually implies a slightly stronger condition:

Theorem 2.2. *For every object X,* $\dim X \geq 1$.

Proof. The dimension $\dim(X)$ is strictly positive and dominates all other eigenvalues of N_X in modulus. Thus, if $\dim X < 1$, we must have N_X nilpotent since N_X^n would tend to 0. But $\dim \mathrm{Hom}(X^{\otimes n}, \mathbf{1}) > 0$ for some n by [12, Lemma F.6], so that N_X^n must be non-zero. □

Finally, we introduce the following standard notation:

- If $X \in \mathcal{C}$ has $\dim(X) = 1$ we say that X is *invertible*. If every simple $X \in \mathcal{C}$ is invertible, we say $\mathcal{C}$ is *pointed*.
- If $X \in \mathcal{C}$ has $\dim(X) \in \mathbb{Z}$ we say that X is *integral*. If every simple $X \in \mathcal{C}$ is integral, we say $\mathcal{C}$ is *integral*.
- If $X \in \mathcal{C}$ is simple and has $\dim(X)^2 \in \mathbb{Z}$ we say that X is *weakly integral*. If every simple $X \in \mathcal{C}$ is weakly integral, we say $\mathcal{C}$ is *weakly integral*. If, moreover, there exists a simple object $X \in \mathcal{C}$ with $\dim(X) \notin \mathbb{Z}$ we say $\mathcal{C}$ is *strictly weakly integral*.

3. Properties Determined by Dimension

In this section we assume that $\mathcal{C}$ is a unitary braided fusion category, and summarize some results showing that several important properties of any anyon are detected by the dimension of the corresponding simple object.

Important examples of unitary braided fusion categories are obtained from quantum groups at roots of unity (see [35] for a survey). Here we briefly outline the construction, mainly for notational purposes.

(1) Let $\mathfrak{g}$ be a simple Lie algebra and $q = e^{\pi i/\ell}$ be a root of unity with $\ell > \check{h}$ (dual Coxeter number of $\mathfrak{g}$) and $\ell \in 2\mathbb{Z}$ for types B, C, F and $\ell \in 3\mathbb{Z}$ for type G.

(2) The representation category of quantum group $U_q\mathfrak{g}$ is a non-semisimple ribbon category. A quotient by the tensor ideal of negligible morphisms (essentially the radical of the trace) of this category gives a braided fusion category $\mathcal{C}(\mathfrak{g}, \ell)$.

(3) The integer $k = (\ell - \check{h})/m$ (where $m = 2$ for types B, C, F and $m = 3$ for type G and $m = 1$ otherwise) is called the *level*, and for classical types A, B, C and D we adopt the alternative notation for $\mathcal{C}(\mathfrak{g}, \ell)$ is $G(N)_k$ for $G = SU, SO$ or Sp of dimension N.

The braid group on n strands $\mathcal{B}_n$ generated by $\sigma_1, \ldots, \sigma_{n-1}$ satisfying

(1) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i \leq n-2$

(2) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$

plays an important role in topological phases of matter: particle exchange induces a representation of $\mathcal{B}_n$ on the state space $\mathrm{Hom}(\mathbf{1}, X^{\otimes n})$ for any object X. More generally we obtain a homomorphism of $\mathcal{B}_n$ into $\mathrm{Aut}(X^{\otimes n})$, which is often lifted to a homomorphism of the group algebra $\rho_X : \mathbb{C}\mathcal{B}_n \to \mathrm{End}(X^{\otimes n})$. Moreover, $\mathcal{H}_n := \bigoplus_i \mathrm{Hom}(X_i, X^{\otimes n})$ is a faithful $\mathrm{End}(X^{\otimes n})$-module and hence we will sometimes abuse notation and regard ρ_X as a $\mathcal{B}_n$-representation on $\mathcal{H}_n$.

3.1. **Abelian Anyons.**

We say that $X \in \mathcal{C}$ (or the corresponding anyon) is *non-abelian* if the $\mathcal{B}_n$-representation $(\rho_X, \mathcal{H}_n)$ has non-abelian image for all $n \geq 3$. This is the first property of $X \in \mathcal{C}$ determined by dimension:

Theorem 3.1 ([39])**.** *If $X \in \mathcal{C}$ is simple and $\dim X > 1$ then X is non-abelian.*

It is clear that any invertible $Z \in \mathcal{C}$ is abelian: $\dim \mathrm{End}(Z^{\otimes n}) = 1$ in this case so $\rho_Z(\mathcal{B}_n)$ acts by scalars. Thus we have a complete characterization of (simple) abelian anyons as those corresponding to invertible objects.

3.2. **Localizable Anyons.**

The $\mathcal{B}_n$ representations ρ_X are somewhat complicated–they exhibit a *hidden locality* [19], but are not explicitly local–rather the action of $\rho_X(\sigma_i)$ act non-trivially on the entire space $\mathcal{H}_n$. Explicitly local representations of $\mathcal{B}_n$ can be obtained from solutions $R \in \mathrm{Aut}(V^{\otimes 2})$ to the Yang-Baxter equation on a vector space V:

$$(R \otimes Id_V)(Id_V \otimes R)(R \otimes Id_V) = (Id_V \otimes R)(R \otimes Id_V)(Id_V \otimes R). \tag{3.1}$$

Such a pair (R, V) is called a *braided vector space*. From a braided vector space we obtain a representation ρ^R of $\mathcal{B}_n$ on $V^{\otimes n}$ via

$$\sigma_i \to Id_V^{\otimes(i-1)} \otimes R \otimes Id_V^{\otimes(n-i-1)}.$$

We say that $X \in \mathcal{C}$ is localizable if there is a braided vector space (R, V) and injective algebra maps $\tau_n : \mathbb{C}\rho_X(\mathcal{B}_n) \to \mathrm{End}(V^{\otimes n})$ such that, for all n, $\rho^R = \tau_n \circ \rho_X$. That is, the following diagram commutes:

$$\begin{tikzcd}
\mathbb{C}\mathcal{B}_n \arrow[d, "\rho_X"] \arrow[dr, "\rho^R"] & \\
\mathbb{C}\rho_X(\mathcal{B}_n) \arrow[r, "\tau_n"] & \mathrm{End}(V^{\otimes n})
\end{tikzcd}$$

Example 3.2. *Consider $\mathcal{C} = SU(2)_2$ an Ising category. There are 3 simple objects $\mathbf{1}, \psi, \sigma$ in $\mathcal{C}$. The object σ has dimension $\sqrt{2}$, and ψ is a (Majorana) fermion. For $R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ the braided vector space $(R, \mathbb{C}^2)$ provides a localization of σ [18].*

Two more slightly less restrictive notions of localizability are studied in [21], namely (k, m)-generalized localizations and quasi-localizations. Under some assumptions, localizability is known to be determined by dimension:

Theorem 3.3 ([38, 21])**.** *Let $X \in \mathcal{C}$ be a simple object. If $\rho_X(\mathbb{C}\mathcal{B}_n) = \mathrm{End}(X^{\otimes n})$ for $n \geq 2$ and X is (generalized or quasi-)localizable then $\dim(X)^2 \in \mathbb{Z}$.*

It is believed that this relationship holds more generally:

Conjecture 3.4 ([38, 21])**.** *Any simple $X \in \mathcal{C}$ is (generalized or quasi-)localizable if, and only if $\dim(X)^2 \in \mathbb{Z}$.*

The Gaussian Yang-Baxter operators described in [23] provide localizations of the generating (fundamental spin) objects in $SO(N)_2$.

3.3. **Universal Anyons.** A given topological model for quantum computation is called (braiding-only) *universal* if any unitary operator can be efficiently approximated up to a phase by braiding anyons [20]. We say that $X \in \mathcal{C}$ is *braiding universal* if $\rho_X(\mathcal{B}_n)$ is dense in $SU(W)$ for each irreducible subrepresentation W of $\mathcal{H}_n$. The first step to verify universality is to check that $\rho_X(\mathcal{B}_n)$ is infinite, and very often (see [20, 27]) this is sufficient. It is therefore of the utmost importance to determine when $\rho_X(\mathcal{B}_n)$ is finite.

Definition 3.5. An object $X \in \mathcal{C}$ has *property F* if the image $\rho_X(\mathcal{B}_n)$ is finite.

A braided fusion category $\mathcal{C}$ has *property F* if the associated braid group representations on the centralizer algebras $\mathrm{End}_{\mathcal{C}}(X^{\otimes n})$ have finite image for all n and all objects X.

Conjecture 3.6. *[31] A braided fusion category $\mathcal{C}$ has property F if, and only if,* $\mathrm{FPdim}(\mathcal{C}) \in \mathbb{Z}$ *(i.e. $\mathcal{C}$ is weakly integral).*

Significant progress has been made towards proving this conjecture:

(1) Every $X \in \mathrm{Rep}(D^\omega(G))$ has property F [17].

(2) Every weakly integral quantum group category has property F, including $SU(2)_2$, $SU(2)_4$, $SU(3)_3$ and $SO(N)_2$ (see [20, 27, 26, 34]).

(3) The standard tensor generators of every non-weakly integral quantum group category do not have property F. (see [18, 27, 37]).

From the discussion above, it is clear that weakly integral braided fusion categories (conjecturally) have interesting properties. The main examples we have of such categories are metaplectic categories, which are closely related to $SO(N)_2$. We will explore these categories in some detail in the next sections.

4. Metaplectic modular categories

Definition 4.1. A **metaplectic modular category (of dimension** $4N$**)** is a modular category $\mathcal{C}$ with positive dimensions that is Grothendieck equivalent to $SO(N)_2$, for some integer $N \geq 2$.

These fusion rules of metaplectic categories differ in important ways depending on the value of $N \pmod 4$. Metaplectic categories for N odd were defined in [25] and studied in [1], while the case $N \equiv 2 \pmod 4$ can be found in [4] where the term "even metaplectic" is used to describe metaplectic modular categories of dimension $4N$ with N even. We will simplify the terminology and call them all metaplectic (of dimension $4N$), and specify the value of $N \pmod 4$ when necessary.

The ubiquity of metaplectic categories among weakly integral modular categories as well as their application to topological phases of matter motivate their classification. The complete classification of weakly integral modular categories of dimension $2^n m$, for $n \in \{0, 1, 2, 3\}$ with m a square-free odd integer has been given, employing the classification of metaplectic modular categories of dimension $4N$ with N odd or $N \equiv 2 \pmod 4$ [6, 4]. In [11] a general description of modular categories of dimension $p^n m$ with m square-free is given in terms of de-equivariantizations. Here we consider metaplectic modular categories of dimension $4N$, recalling the known results for $4 \nmid N$ with new results in the case $4 \mid N$.

A significant role is played by pointed modular categories, i.e. modular categories with only invertible simple objects. The classification of pointed modular categories is well-known (going back essentially to [13], also see [12]): they correspond to pairs (A, q) where A is a finite abelian group and q is a non-degenerate quadratic form $q : A \to \mathbb{Q}/\mathbb{Z}$ i.e. $q(-a) = q(a)$ and the symmetric bilinear form on A defined by $\sigma(a, b) = q(a+b) - q(a) - q(b)$ is non-degenerate. We denote such a category by $\mathcal{C}(A, q)$.

Two processes that we employ in our analysis are gauging and de-gauging. First let us describe de-gauging. Let $\mathcal{C}$ be modular and $\text{Rep}(G) \cong \mathcal{D} \subset \mathcal{C}$ a Tannakian subcategory (here a Tannakian category is a symmetric braided fusion category equivalent to $\text{Rep}(G)$ for some finite group G). The G-de-equivariantization $\mathcal{C}_G$ of $\mathcal{C}$ is a faithfully G-graded category (in fact, a braided G-crossed category) with trivial component $[\mathcal{C}_G]_e$ a modular category of dimension $\dim(\mathcal{C})/|G|^2$ (see [10]). $[\mathcal{C}_G]_e$ is the G-de-gauging of $\mathcal{C}$. The reverse process, G-gauging, is more complicated. Here one starts with a modular category $\mathcal{B}$ and an action of a finite group G by braided tensor autoequivalences: $\rho : G \to \mathrm{Aut}_{\otimes}^{br}(\mathcal{B})$. A G-gauging of $\mathcal{B}$, when it exists, is a new modular category obtained by first constructing a G-graded fusion category $\mathcal{D}$ with trivial component $\mathcal{D}_e = \mathcal{B}$ and then equivariantizing $\mathcal{D}^G$. There are obstructions to the existence of a gauging, and when the obstructions vanish there can be many G-gaugings.

Here is a key example: consider the pointed modular category $\mathcal{C}(\mathbb{Z}_N, q)$. The elements of the group $\mathrm{Aut}_{\otimes}^{br}(\mathcal{C}(\mathbb{Z}_N, q))$ are simply those $\phi \in \mathrm{Aut}(\mathbb{Z}_N)$ that preserve q. The *particle-hole symmetry* $\phi : a \mapsto -a$ is an example of such an automorphism. Furthermore, it can be shown that the obstructions vanish in this case, so the action $\rho : \mathbb{Z}_2 \to \mathrm{Aut}_{\otimes}^{br}(\mathcal{C}(\mathbb{Z}_N, q))$ defined by $\rho(1) = \phi$ can be gauged.

4.1. **Metaplectic modular categories of dimension $4N$ with $4 \nmid N$.** The metaplectic modular categories of dimension $4N$ with $4 \nmid N$ have been studied in [1, 4]. Most of the fusion rules for such a category were given in [31] with more complete details in [25, 4]. For N odd there are 2 invertible objects, $\frac{N-1}{2}$ simple objects of dimension 2 and 2 simple objects of dimension $\sqrt{N}$. All simple objects are self-dual in this case. In the case $N \equiv 2 \pmod 4$ we have 4 invertible objects (including one pair of non-self-dual objects), 4 (non-self-dual) simple objects of dimension $\sqrt{N/2}$, and $\frac{N}{2} - 1$ simple objects of dimension 2 (all of which are self-dual).

We have a complete classification of these categories:

Theorem 4.2. *[1, 4] If $\mathcal{C}$ is a metaplectic category of dimension $4N$, with $4 \nmid N$, then $\mathcal{C}$ is a gauging of the particle-hole symmetry of a $\mathbb{Z}_N$-cyclic modular category $\mathcal{C}(\mathbb{Z}_N, q)$. Moreover, for $N = p_1^{k_1} \cdots p_r^{k_r}$, with p_i distinct primes (where if $p_1 = 2, k_1 = 1$), there are exactly 2^{r+1} many inequivalent such metaplectic modular categories.*

4.2. **Metaplectic categories of dimension $4N$, with $4 \mid N$.** The modular category $SO(N)_2$ with $N \equiv 0 \pmod 4$ corresponds to Lie type D_k with $2k = N$, has rank $k+7$ and dimension $4N$ [31]. The simple objects have dimension $1, 2$ and $\sqrt{k}$.

The simple objects of $SO(N)_2$ are naturally labeled by $\mathfrak{so}_N$ weights μ with $\mu^1 + \mu^2 \leq 2$, i.e. the sum of the first two coordinates can be at most 2. We will provide them with less cumbersome labels, after identifying their weights in terms of the fundamental weights $\lambda_j = (1, \ldots, 1, 0, \ldots 0)$ with j 1s for $j \leq k-2$, $\lambda_{k-1} = \frac{1}{2}(1, \ldots, 1, -1)$, and $\lambda_k = \frac{1}{2}(1, \ldots, 1)$. Setting $r = \frac{k}{2} - 1$, we have simple objects as follows:

- objects with weights $\mathbf{0}, 2\lambda_1, 2\lambda_{k-1}, 2\lambda_k$ will be denoted $\mathbf{1}, fg, f, g$,
- simple objects with weights $\lambda_2, \ldots, \lambda_{k-2}$ will be denoted $X_0, \ldots, X_{r-1}$,
- simple objects with weights $\lambda_1, \lambda_3, \ldots, \lambda_{k-3}, \lambda_{k-1} + \lambda_k$ will be denoted $Y_0, \ldots, Y_r$,

- simple objects with weights $\lambda_{k-1}, \lambda_1 + \lambda_k$ will be denoted V_1, V_2, and
- simple objects with weights $\lambda_k, \lambda_1 + \lambda_{k-1}$ will be denoted W_1, W_2.

All simple objects are self-dual, and the X_i and Y_i have dimension 2, while V_i, W_i have dimension $\sqrt{k}$. For $k > 2$ the key fusion rules are as follows, where we abuse notation and write $=$ for $\cong$:

- $f^{\otimes 2} = g^{\otimes 2} = \mathbf{1}$, $f \otimes X_i = g \otimes X_i = X_{r-i-1}$ and $f \otimes Y_i = g \otimes Y_i = Y_{r-i}$
- $g \otimes V_1 = V_2, f \otimes V_1 = V_1$ and $f \otimes W_1 = W_2, g \otimes W_1 = W_1$
- $V_1^{\otimes 2} = \mathbf{1} \oplus f \oplus \bigoplus_{i=0}^{r-1} X_i$
- $W_1^{\otimes 2} = \mathbf{1} \oplus g \oplus \bigoplus_{i=0}^{r-1} X_i$
- $W_1 \otimes V_1 = \bigoplus_{i=0}^{r} Y_i$
- $X_i \otimes X_j = \begin{cases} X_{i+j+1} \oplus X_{j-i-1} & i < j \leq \frac{r-1}{2} \\ \mathbf{1} \oplus fg \oplus X_{2i+1} & i = j < \frac{r-1}{2} \\ \mathbf{1} \oplus f \oplus g \oplus fg & i = j = \frac{r-1}{2} < r-1 \end{cases}$
- $Y_i \otimes Y_j = \begin{cases} X_{i+j} \oplus X_{j-i-1} & i < j \leq \frac{r}{2} \\ \mathbf{1} \oplus fg \oplus X_{2i} & i = j \leq \frac{r-1}{2} \\ \mathbf{1} \oplus f \oplus g \oplus fg & i = j = \frac{r}{2}. \end{cases}$

Notice that all other fusion rules may be easily derived from the above by tensoring with f or g as needed. For example $V_1 \otimes V_2 = g \otimes V_1^{\otimes 2} = f \oplus fg \oplus \bigoplus_{i=0}^{r-1} X_i$.

For $k = 2$, i.e. $SO(4)_2$ we have 9 simple objects: $\mathbf{1}, f, g, fg, Y_0, V_1, V_2, W_1, W_2$ and the fusion rules are the same as $SU(2)_2 \boxtimes SU(2)_2$. The applicable fusion rules above still hold. Indeed, by [12, Corollary B.12] any such category is equivalent to a Deligne product of Ising-type modular categories (of which there are 8).

Recall that a metaplectic category of dimension $4N$ with $4 \mid N$ is any modular category $\mathcal{C}$ with the same fusion rules as above. Fix such a category $\mathcal{C}$ and label the simple objects as above. The objects $\mathbf{1}, f, g$ and fg are invertible, and their isomorphism classes form the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ under tensor product. In particular the universal grading group $\mathcal{U}(\mathcal{C}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ with graded components labeled as follows:

(1) The adjoint category $\mathcal{C}_0$: with simple objects $\mathbf{1}, fg, f, g$ and $X_0, \ldots, X_{r-1}$

(2) $\mathcal{C}_{(1,1)}$ with simple objects $Y_0, \ldots, Y_r$

(3) $\mathcal{C}_{(1,0)}$ with simple objects V_1, V_2

(4) $\mathcal{C}_{(0,1)}$ with simple objects W_1, W_2.

Lemma 4.3. *Let $\mathcal{C}$ be a metaplectic modular category of dimension $4N$ with $4 \mid N$, with simple objects labeled as above. Then:*

(1) fg centralizes every object in $\mathcal{C}_0 \oplus \mathcal{C}_{(1,1)}$.

(2) fg is a boson, i.e. $\theta_{fg} = 1$.

Proof. Observe that since the pointed subcategory $\mathcal{C}_{pt}$ with simple objects $\mathbf{1}, f, g, fg$ is a subcategory of $\mathcal{C}_{ad} = \mathcal{C}_0$ and $\mathcal{C}_{pt} = \mathcal{C}'_{ad}$ we see that $\mathcal{C}_{pt}$ is symmetric, i.e. self-centralizing. Moreover, since $fg \in \mathcal{C}_{pt} = \mathcal{C}'_{ad}$ it is clear that fg centralizes $\mathcal{C}_0$. For $Y_i \in \mathcal{C}_{(1,1)}$ we have $fg \otimes Y_i = Y_i$ and the balancing equation gives:

$$\theta_{Y_i}\theta_{fg}S_{Y_i,fg} = \theta_{Y_i}\dim(Y_i)$$

so that $S_{Y_i,fg} = \dim(fg)\dim(Y_i)$ if and only if $\theta_{fg} = 1$. If $k > 2$ (so $r > 0$) there is a 2-dimensional object $X_0 \in \mathcal{C}_{ad}$ with $X_0 \otimes fg = X_0$ and $S_{X_0,fg} = 2 = \dim(X_0)\dim(fg)$, so that the balancing equation implies $\theta_{fg} = 1$, and the result follows.

For the case $k = 2$, $\mathcal{C}$ is a product of Ising-type categories, and there is a labeling ambiguity among f, g and fg. Precisely one of these is a boson, since the non-trivial invertible object in any Ising category is a fermion. So we may assume fg is the boson, and the result follows as above. $\square$

In particular, the subcategory $\langle fg\rangle$ generated by fg is equivalent, as a symmetric ribbon category, to $\mathrm{Rep}(\mathbb{Z}_2)$.

We can now prove a partial analogue to Theorem 4.2 for metaplectic modular categories of dimension $4N$ with $4 \mid N$. One exception is the case $N = 4$.

Theorem 4.4. *If $\mathcal{C}$ is a metaplectic modular category of dimension $4N > 16$ with $4 \mid N$ then the de-equivariantization $\mathcal{D} := \mathcal{C}_{\mathbb{Z}_2}$ by $\langle fg\rangle = \mathrm{Rep}(\mathbb{Z}_2)$ is a generalized Tambara-Yamagami category of dimension $4N$, and, the trivial component $\mathcal{D}_0 := [\mathcal{C}_{\mathbb{Z}_2}]_0 \cong \mathcal{C}(\mathbb{Z}_N, q)$ is a pointed cyclic modular category. Moreover, $\mathcal{C}$ is obtained from $\mathcal{C}(\mathbb{Z}_N, q)$ via a $\mathbb{Z}_2$-gauging of the particle-hole symmetry.*

Proof. We continue with the notation and labeling as above, with $\mathbf{1}, f, g, fg, X_i, Y_j$ with $0 \le i \le r-1$ and $0 \le j \le r$ where $r = \frac{k}{2} - 1$ and $N = 2k$.

As we noted previously, by Lemma 4.3, $\langle gf\rangle \cong \mathrm{Rep}\,(\mathbb{Z}_2)$ is a Tannakian subcategory of $\mathcal{C}$. In particular we have the de-equivariantization functor $F : \mathcal{C} \to \mathcal{C}_{\mathbb{Z}_2}$, the image of which is a braided $\mathbb{Z}_2$-crossed fusion category of dimension $2N$ [12]. In particular, the trivial component $\mathcal{D}_0$ of $\mathcal{D} = \mathcal{C}_{\mathbb{Z}_2}$ is modular of dimension N [12, Proposition 4.56(ii)].

Since gf fixes the 2-dimensional objects X_i, Y_i of $\mathcal{C}$, their images under F give rise to $k-1$ inequivalent pairs $X_i^{(1)}, X_i^{(2)}, Y_i^{(1)}, Y_i^{(2)}$ of distinct invertible objects in $\mathcal{D}$. On the other hand, $gf \otimes \mathbf{1} = gf$ and $g \otimes gf = f$, so that $F(gf) = F(\mathbf{1}) = \mathbf{1}_{\mathcal{D}}$ and $F(g) = F(f) = Z$ are distinct invertible objects in $\mathcal{D}$. In all, we have $2(k-1) + 2 = N$ invertible objects, all of which are in the trivial component $\mathcal{D}_0$. Thus $\mathcal{D}_0 \cong \mathcal{C}(A, q)$ for some abelian group of order N.

Since the 4 objects in $\mathcal{C}_{(0,1)} \cup \mathcal{C}_{(1,0)}$ form two orbits under $\_\otimes fg$, we see that $\mathcal{D}_1$ has two non-integral simple objects which each have dimension $\sqrt{N/2}$. In particular, $\mathcal{D}$ is generalized Tambara-Yamagami category [28].

It remains to verify that the classes of simple objects in $\mathcal{D}_0$ form a *cyclic* group, which we do following the inductive idea of [1].

First note that $Y_0 \otimes Y_0 = \mathbf{1} \oplus fg \oplus X_0$. Since $F(\mathbf{1}) = F(fg) = \mathbf{1}_{\mathcal{D}}$ is the the trivial object under the de-equivariantization, we have:

$$F(Y_0^{\otimes 2}) = \left(Y_0^{(1)} \oplus Y_0^{(2)}\right)^{\otimes 2} = 2(Y_0^{(1)} \otimes Y_0^{(2)}) \oplus \left(Y_0^{(1)}\right)^{\otimes 2} \oplus \left(Y_0^{(2)}\right)^{\otimes 2} = \mathbf{1}_{\mathcal{D}} \oplus X_0^{(1)} \oplus X_0^{(2)}$$

We will show that the class of $Y_0^{(1)}$ generates $[\mathcal{C}_{\mathbb{Z}_2}]_0$. Examining multiplicities we see that we may assume (using the labeling ambiguity $X_0^{(j)}$, $j = 1, 2$) $Y_0^{(1)} \otimes Y_0^{(2)} = \mathbf{1}_{\mathcal{D}}$ while $Y_0^{(j)} \otimes Y_0^{(j)} = X_0^{(j)}$.

Proceeding in a similar way with $Y_0 \otimes X_0 = Y_0 \oplus Y_1$ we must match the 4 simple objects $Y_0^{(i)} \otimes X_0^{(j)}$ for $1 \leq i, j \leq 2$ with the four simple objects $Y_a^{(b)}$ for $a = 0, 1$ and $b = 1, 2$. Now since $Y_0^{(2)} = (Y_0^{(1)})^*$ we must have

$$Y_0^{(1)} \otimes X_0^{(1)} \oplus Y_0^{(2)} \otimes X_0^{(2)} = Y_1^{(1)} \oplus Y_1^{(2)}.$$

This is again a labeling ambiguity so we may define, without loss of generality, $Y_0^{(j)} \otimes X_0^{(j)} = Y_1^{(j)}$ for $j = 1, 2$.

Now notice that for $n \leq 2r$ the tensor power $Y_0^{\otimes n}$ contains exactly one simple object that has not appeared in lower tensor powers: namely $X_i \subset Y_0^{\otimes(2i+2)}$ and $Y_i \subset Y_0^{\otimes(2i+1)}$. Thus we may proceed inductively and define (using Frobenius reciprocity and the labeling ambiguity) for $j = 1, 2$ and $0 \leq i \leq r - 1$:

$$Y_{i+1}^{(j)} := Y_0^{(j)} \otimes X_i^{(j)}, \quad X_i^{(j)} := Y_0^{(j)} \otimes Y_i^{(j)}.$$

Thus we see that all $Y_i^{(1)}$ and $X_i^{(1)}$ are tensor powers of $Y_0^{(1)}$ and all $X_i^{(2)}$ and $Y_i^{(2)}$ are tensor powers of $Y_0^{(2)}$. Next, note that $Y_i^{(2)}$ represents the isomorphism class that is the multiplicative inverse to that of $Y_i^{(1)}$ in the Grothendieck ring of $[\mathcal{C}_{\mathbb{Z}_2}]_0$, since Y_j is self-dual. Thus all $Y_a^{(j)}$ for $0 \leq a \leq r$ and $X_b^{(j)}$ for $0 \leq b \leq r - 1$ ($j = 1, 2$) are in the subcategory generated by $Y_0^{(1)}$.

It remains to show that $F(f) = F(g) = Z$ is a tensor power of $Y_0^{(1)}$. For this we compute:

$$Y_0 \otimes Y_r = Y_0 \otimes Y_0 \otimes f = (\mathbf{1} \oplus fg \oplus X_0) \otimes f = X_{r-1} \oplus f \oplus g$$

Applying the functor F and the assignments above we see that $Y_0^{(j)} \otimes Y_r^{(j)} = X_{r-1}^{(\bar{j})}$ for $j = 1, 2$ where $\bar{2} = 1$ and $\bar{1} = 2$. This leaves, for $j = 1, 2$, $Y_0^{(j)} \otimes Y_r^{(\bar{j})} = Z$. In particular, since $Y_r^{(2)}$ is a tensor power of $Y_0^{(1)}$, we have that Z is also. Thus $\mathcal{D}_0 \cong \mathcal{C}(\mathbb{Z}_N, q)$ is a pointed cyclic modular category.

Now we apply the same argument as in [1] and [4] to see that the only $\mathbb{Z}_2$ gauging on $\mathcal{C}(\mathbb{Z}_N, q)$ that has four invertible objects is the particle-hole symmetry, i.e. inversion on $\mathbb{Z}_N \cong \mathbb{Z}_{2^a} \times \mathbb{Z}_{p_1^{a_1}} \times \cdots \times \mathbb{Z}_{p_b^{a_b}}$ (where $a \geq 2$. Indeed, there can be only two fixed points under the $\mathbb{Z}_2$ action (namely the identity and the unique element of order 2 coming from the $\mathbb{Z}_{2^a}$ factor) as otherwise we obtain $\boxtimes$ factors of the form $\mathcal{C}(\mathbb{Z}_{p_i^{a_i}}, q_i)$ in the gauged category, which is incompatible with the structure of $\mathcal{C}$. □

Remark 4.5. Notice that this argument fails for $N = 4$. As we have observed above, metaplectic categories of dimension 16 are of the form $\text{Ising}^{(\nu_1)} \boxtimes \text{Ising}^{(\nu_2)}$ where $\nu_i \in \mathbb{Z}_{16}^*$. In that case we have $Y_0^{\otimes 2} = \mathbf{1} \oplus f \oplus g \oplus fg$, so that we cannot determine $Y_0^{(1)} \otimes Y_0^{(2)}$ from $(Y_0^{(1)})^{\otimes 2}$ using only the fusion rules–one must be $\mathbf{1}_{\mathcal{D}}$ and the other Z. The two possiblities lead to $\mathcal{D}_0 \cong \mathcal{C}(\mathbb{Z}_4, q_1)$ or $\mathcal{D}_0 \cong \mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2, q_2)$. Both of these can occur: see [3].

Lemma 4.6. *Consider $\mathcal{C}$ a metaplectic modular category of dimension $4N$ with $4 \mod N$. If $8 \mid N$ then all the non-trivial invertible objects of $\mathcal{C}$ are bosons, and if $8 \nmid N$ two are fermions (and one is a boson).*

Proof. From the description above, we know that there are $r = \frac{N}{4} - 1$ 2-dimensional simple objects in the adjoint component $\mathcal{C}_0$, and $r + 1$ 2-dimensional simple objects in $\mathcal{C}_{(1,1)}$. Moreover, the pointed subcategory is symmetric and has fusion rules like $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Since $\mathcal{C}_0' = \mathcal{C}_{pt}$, if one of the invertibles is a fermion, the action of tensoring with that fermion must be fixed-point free on $\mathcal{C}_0$. Then, the number of 2-dimensional simple objects in $\mathcal{C}_0$, r, must be even in this case. In particular, $N = \dim \mathcal{C} = 4 + 4r$ is not divisible by 8. So if $8 \mid N$, r is odd and so there are no fermions.

On the other hand, if $8 \nmid N$ we must have r even, and so $Y_{\frac{r}{2}}^{\otimes 2} = \mathbf{1} \oplus f \oplus g \oplus fg$. The balancing equation then gives, for example, $-2\theta_{Y_{\frac{r}{2}}} \theta_f = 2\theta_{Y_{\frac{r}{2}}}$, so that $\theta_f = \theta_g = -1$. □

We would like to provide a careful count of the inequivalent metaplectic modular categories $\mathcal{C}$ of dimension $4N$ with $4 \mid N$, in analogy with Theorem 4.2. As our argument is only a sketch, we do not present it as a theorem. For $N \geq 8$ we know from the above that $\mathcal{C}$ is obtained from $\mathcal{C}(\mathbb{Z}_N, q)$ for some q via particle-hole gauging. Factoring $N = 2^a p_1^{a_1} \cdots \pi_b^{a_b}$ as above ($a \geq 2$) we find that there are exactly $4 \cdot 2^b$ inequivalent categories of the form $\mathcal{C}(\mathbb{Z}_N, q)$ (see [22]). We wish to determine the number of distinct (particle-hole) gaugings of a given fixed $\mathcal{C}(\mathbb{Z}_N, q)$.

Let $\rho : \mathbb{Z}_2 \to \text{Aut}(\mathbb{Z}_N)$ denote the map determined by $\rho(1)(n) = -n$, i.e. the particle-hole symmetry. One known extension $\mathcal{D}$ of $\mathcal{C}(\mathbb{Z}_N, q)$ by ρ has two defects $\sigma_\pm$ with fusion rules determined by

- $\sigma_+ \otimes \sigma_+ = \bigoplus_{a \text{ even}} [a]_N$

- $\sigma_+ \otimes \sigma_- = \bigoplus_{a \text{ odd}} [a]_N$
- $\sigma_\pm \otimes [a] = \begin{cases} \sigma_\pm & [a]_N \text{ even} \\ \sigma_\mp & [a]_N \text{ odd} \end{cases},$

where $[a]_N$ denotes the simple object corresponding to $a \in \mathbb{Z}_N$ [3].

Given a fixed extension (in our case, $\mathcal{D}$), any extension of $\mathcal{C}(\mathbb{Z}_N, q)$ by ρ corresponds to a gauging datum, i.e. a pair $(\alpha, \beta) \in H^2_\rho(\mathbb{Z}_2, \mathbb{Z}_N) \times H^3(\mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ such that a certain obstruction $O_4(\rho, \alpha)$ vanishes [9].

At the level of fusion rules, the action of the nontrivial element $\alpha \in H^2_\rho(\mathbb{Z}_2, \mathbb{Z}_N)$ twists the tensor product of defects by a representative 2-cocycle. One such representative cocycle is the normalized cocycle $\omega \in Z^2(\mathbb{Z}_2, \mathbb{Z}_N)$ determined by $\omega(1,1) = N/2$. The new tensor product $\otimes'$ is given by

$$x_g \otimes' x_h = [\omega(g,h)]_N \otimes x_g \otimes x_h,$$

for x_g in the g-component of the extension and x_h in the h-component [14]. Since ω is normalized, the only non-trivial twisting occurs when both g and h are nontrivial. Since $\omega(1,1) = N/2$ is even (because 4 divides N),

$$\sigma_\pm \otimes' \sigma_\pm = [N/2]_N \otimes \sigma_\pm \otimes \sigma_\pm = \bigoplus_{a \text{ even}} [a]_N = \sigma_\pm \otimes' \sigma_\pm$$

and

$$\sigma_\pm \otimes' \sigma_\mp = [N/2]_N \otimes \sigma_\pm \otimes \sigma_\mp = \bigoplus_{a \text{ odd}} [a]_N = \sigma_\pm \otimes' \sigma_\mp.$$

Hence, the action of α on the fusion rules is trivial. However, since β is an element of a torsor over $H^2_\rho(\mathbb{Z}_2, \mathbb{Z}_N)$, the choice of α is still relevant.

There are two potential obstructions to extending $\mathcal{C}(\mathbb{Z}_N, q)$ by ρ. The first obstruction in $H^3_\rho(\mathbb{Z}_2, \mathbb{Z}_N)$ vanishes since we know that a gauging exists. The second obstruction vanishes since $H^4(\mathbb{Z}_2, U(1)) \cong 0$. There is only one possible set of fusion rules on the $\mathbb{Z}_2$-extension since the action of $H^2_\rho(\mathbb{Z}_2, \mathbb{Z}_N)$ on the fusion rules is trivial. There is a choice of associativity constraints on the $\mathbb{Z}_2$-extension, so that *a priori* we have 4 distinct theories. However, by examining the pairs of FS-indicators of the defects $\sigma_\pm$ as in [3, Section X.F] we find that they are $(1,1), (-1,-1), (1,-1)$ and $(-1,1)$, so that the latter two theories can be identified by relabeling. Thus we have 3 distinct gaugings for each fixed $\mathcal{C}(\mathbb{Z}_N, q)$, yielding $3(2^{b+2})$ metaplectic modular categories of dimension $4N > 16$ with $4 \mid N$.

4.3. **Degenerate case: metaplectic modular categories of dimension** 16. As we observed above, any metaplectic modular category of dimension 16 has fusion rules like $SU(2)_2 \times SU(2)_2$, i.e. a product of Ising categories.

Question 4.7. *How many inequivalent modular categories of the form* $\mathrm{Ising}^{\nu_1} \boxtimes \mathrm{Ising}^{\nu_2}$ *exist?*

Notice that when $N = 4$ the argument above is still valid if applied only to $\mathcal{C}(\mathbb{Z}_4, q)$: we still obtain 12 distinct theories of the form $\mathrm{Ising}^{\nu_1} \boxtimes \mathrm{Ising}^{\nu_2}$ from each of these. There are 8 more $\mathrm{Ising}^{\nu_1} \boxtimes \mathrm{Ising}^{\nu_2}$ theories: two from each of 4 distinct $\mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2, q)$ modular categories containing a fermion. These are usually called TC (Toric Code), 3F (3 fermions), Sem^2 (semion-squared) and $\overline{\mathrm{Sem}}^2$ (semion-conjugate-squared). Thus we expect a total of 20 metaplectic modular categories of dimension 16.

This expectation is also supported by a count of T-matrices up to dimension-preserving permutation. To make this count, we use the following table of dimensions and twists of simple objects of $\mathrm{Ising}^{\nu_1} \boxtimes \mathrm{Ising}^{\nu_2}$:

$(\dim_{X\boxtimes Y}, \theta_{X\boxtimes Y})$	$\mathbf{1}$	ψ	σ
$\mathbf{1}$	$(1,1)$	$(1,-1)$	$(\sqrt{2}, e^{\frac{\pi i \nu_2}{8}})$
ψ	$(1,-1)$	$(1,1)$	$(\sqrt{2}, -e^{\frac{\pi i \nu_2}{8}})$
σ	$(\sqrt{2}, e^{\frac{\pi i \nu_1}{8}})$	$(\sqrt{2}, -e^{\frac{\pi i \nu_1}{8}})$	$(2, e^{\frac{\pi i(\nu_1+\nu_2)}{8}})$

We claim that any equivalence of modular categories $F : \mathrm{Ising}^{\nu_1} \boxtimes \mathrm{Ising}^{\nu_2} \to \mathrm{Ising}^{\nu_1'} \boxtimes \mathrm{Ising}^{\nu_2'}$ is determined by $F(\sigma \boxtimes \mathbf{1})$. Since F must preserve simple objects of dimension $\sqrt{2}$, we have $F(\sigma \boxtimes \mathbf{1}) \in \{\sigma \boxtimes \mathbf{1}, \mathbf{1} \boxtimes \sigma, \psi \boxtimes \sigma, \sigma \boxtimes \psi\}$. Thus, $F((\psi \oplus \mathbf{1}) \boxtimes \mathbf{1}) = F(\sigma^{\otimes 2} \boxtimes \mathbf{1}) = (F(\sigma \boxtimes \mathbf{1}))^{\otimes 2} \in \{(\psi \oplus \mathbf{1}) \boxtimes \mathbf{1}, \mathbf{1} \boxtimes (\psi \oplus \mathbf{1})\}$. Hence, $F(\sigma \boxtimes \mathbf{1})$ determines $F(\psi \boxtimes \mathbf{1}) \in \{\psi \boxtimes \mathbf{1}, \mathbf{1} \boxtimes \psi\}$. The value of $F(\psi \boxtimes \mathbf{1})$ also determines $F(\mathbf{1} \boxtimes \psi)$ since these are the only fermions. Hence, $F(\sigma \boxtimes \mathbf{1})$ also determines $F(\sigma \boxtimes \psi)$. Since F must preserve the unique simple object of dimension 2, we have $\nu_1' + \nu_2' = \nu_1 + \nu_2 \pmod{16}$. Since the twist of $F(\sigma \boxtimes \mathbf{1})$ is $\nu_1 \in \{\nu_i', \nu_i' + 8\}$ for some $i \in \{1, 2\}$, this equation determines $\nu_2 \in \{\nu_{3-i}', \nu_{3-i}' + 8\}$, hence the value of $F(\mathbf{1} \boxtimes \sigma)$. The other simple objects are generated by these values, proving the claim.

We have the following count of T-matrices up to dimension-preserving permutation. There are $8 \cdot 8 = 64$ ordered pairs (ν_1, ν_2). This splits into 56 pairs of distinct (ν_1, ν_2) and 8 pairs of the form (ν, ν). Since ν is odd, the orbit of (ν, ν) under dimension- and twist-preserving permutation is $\{(\nu, \nu), (\nu + 8, \nu + 8)\}$ of order 2. The 56 pairs of distinct (ν_1, ν_2) further break up into 8 pairs such that $\nu_1 + 8 = \nu_2$ and 48 pairs such that $\nu_1 + 8 \neq \nu_2$. In the case that $\nu_1 + 8 = \nu_2$, the orbit of (ν_1, ν_2) is $\{(\nu_1, \nu_2), (\nu_2, \nu_1)\}$ of order 2. In the case that $\nu_1 + 8 \neq \nu_2$, the orbit is $\{(\nu_1, \nu_2), (\nu_2, \nu_1), (\nu_1 + 8, \nu_2 + 8), (\nu_2 + 8, \nu_1 + 8)\}$ of order 4. Thus, we have $8/2 + 8/2 + 48/4 = 20$ distinct T-matrices up to dimension-preserving permutation.

More generally, a lower bound on the number of inequivalent metaplectic modular categories could be computed by determining the modular data:

Question 4.8. *What are the twists of each metaplectic modular category of dimension $4N$ in terms of the twists of the pointed modular category $\mathcal{C}(\mathbb{Z}_N, q)$?*

Formula (412) of [9] conjecturally relates twists in the gauged theory to twists in the extension. This still leaves the problem of finding the twists of the defects in the extension. As far as we can tell, the only way to do this is to solve the G-crossed heptagon equations.

5. Modular categories of dimension $16m$, with m odd square-free integer

In this section we will consider $\mathcal{C}$ a modular category of dimension $16m$, with m an odd square-free integer. We have seen examples, namely metaplectic modular categories of dimension $4N$ with $N = 4m$ with m square-free and odd. The goal is to give a classification of this class of categories similar to the ones in [6, 4] of modular categories of dimension $4m$ and $8m$, respectively. In what follows, we give a classification of modular categories of dimension $16m$ under certain restrictions.

From [4, Lemma 4.1], we know that if 16 divides the order of the universal grading group $U(\mathcal{C})$ of $\mathcal{C}$, then $\mathcal{C}$ is pointed. So we can assume that $16 \nmid |U(\mathcal{C})|$.

The only *integral* metaplectic modular categories of dimension $16m$ with m square-free are those with $m = 1$. We will disregard this case, pausing only to ask:

Question 5.1. *Are integral modular categories of dimension* $16m$ *with* $m > 1$ *an odd square-free integer pointed?*

Remark 5.2. While *integral* modular categories of dimension 16 are pointed ([4, Lemma 4.3]), it is not the case that all integral modular categories of dimension 2^k are pointed. For example $SO(8)_2$ is a non-pointed integral modular category of dimension 32, which has 4 invertible objects and 7 simple objects of dimension 2.

Now, we will assume that $\mathcal{C}$ is a strictly weakly integral category of dimension $16m$ with m an odd square-free integer. Moreover, we can consider $\mathcal{C}$ *prime* (i.e. not of the form $\mathcal{C}_1 \boxtimes \mathcal{C}_2$) since otherwise it reduces to known cases, see [6, 4].

Prime modular categories of dimension $4m$ and $8m$ essentially arise from metaplectic categories when they are not pointed. Thus it is natural to ask:

Question 5.3. *Are prime strictly weakly integral modular categories of dimension* $16m$ *for* $m > 1$ *odd and square-free Grothendieck equivalent to* $\mathrm{SO}\,(4m)_2$*?*

Strictly weakly integral modular categories of dimension 16 were classified in [4, Lemma 4.9]. These categories are a Deligne product of an Ising category and a pointed modular category, i.e. a Generalized Tamabra-Yamagami category.

As in the proof of Theorem 3.1 in [6], using the de-equivariantization process, we can assume that $|U(\mathcal{C})| = 2^k$, for k an integer number.

We will not provide a complete answer to Question 5.3, but we will give a partial response and pose some general questions arising from our analysis.

Remark 5.4. Let $\mathcal{C}$ be a modular category of dimension $2^n m$, with m an odd square-free integer. Then, the possible dimensions of integral simple objects of $\mathcal{C}$ are 2^k, with $k = 0, \ldots, \lfloor \frac{n}{2} \rfloor$ (see e.g. [11, Lemma 5.2], [15, Theorem 2.11 (i)]).

Remark 5.5. Let $\mathcal{C}$ be a fusion category and $X \in \mathcal{C}$. Consider $G[X] = \{Y \in \mathcal{C} \mid Y \otimes X \cong X\}$. If $Y \in G[X]$, then $\operatorname{FPdim} Y = 1$.

Moreover, $X \otimes X^* = \sum_{Y \in G[X]} Y + \sum_{\operatorname{FPdim} Z > 1} N_{X,X^*}^Z Z$.

Lemma 5.6. *Let $\mathcal{C}$ be a strictly weakly integral modular categories of dimension $2^n m$, with m an odd square-free integer and $n \geq 3$. Then the universal grading group $\mathcal{U}(\mathcal{C})$ has $|\mathcal{U}(\mathcal{C})| \geq 4$.*

Proof. Since $\mathcal{U}(\mathcal{C})$ is non-trivial and the GN grading group is also non-trivial, it is enough to show that $\mathcal{U}(\mathcal{C}) \ncong \mathbb{Z}_2$. Suppose that $\mathcal{C}$ is as in the statement with $\mathcal{U}(\mathcal{C}) \cong \mathbb{Z}_2$. Notice that the GN-grading group is also $\mathbb{Z}_2$. By faithfulness of the universal grading we have $\dim \mathcal{C}_{ad} = 2^{n-1} m$ (where $n \geq 3$). From the definition of dimension and Remark 5.4, we have $2^{n-1} m = 2 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} 2^{2k} a_k$, where a_k is the number of 2^k-dimensional simple objects in $\mathcal{C}_{ad}$. Since $n > 3$ and $4 \nmid 2$ we have a contradiction. □

Lemma 5.7. *Let m be an odd square-free integer and $n \geq 3$. There are no prime self-dual strictly weakly integral modular categories of dimension $2^n m$ whose universal grading group has order 2^{n-1}.*

Proof. By contradiction, assume that such a category exists. Since $\mathcal{C}$ is self-dual, we must have that $\mathcal{U}(\mathcal{C}) \cong \mathbb{Z}_2^{n-1}$. So let $\mathcal{C}_g$ be a non-integral component. Then $\mathcal{C}_g \otimes \mathcal{C}_g \subset \mathcal{C}_{ad}$. Thus $\mathcal{D} := \mathcal{C}_{ad} \oplus \mathcal{C}_g$ is a fusion category of dimension $4m$. Taking centralizers we have $\mathcal{Z}_{\mathcal{D}}(\mathcal{D}) \subset \mathcal{Z}_{\mathcal{D}}(\mathcal{C}_{ad}) = (\mathcal{C}_{ad})_{pt}$.

Since $\mathcal{C}_{ad}$ has dimension $2m$, we have $2m = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{2k} a_k$, where a_k is the number of 2^k-dimensional simple objects in $\mathcal{C}_{ad}$, by Remark 5.4. Thus $a_0 = 2$ because m is odd. So either $\mathcal{Z}_{\mathcal{D}}(\mathcal{C}_{ad} \oplus \mathcal{C}_g) \cong \operatorname{Vec}$, sVec, or $\operatorname{Rep}(\mathbb{Z}_2)$.

The case $\mathcal{Z}_{\mathcal{D}}(\mathcal{D}) \cong \operatorname{Vec}$ is not possible as $\mathcal{C}$ is prime. Since there are only 2 invertible objects in $\mathcal{C}_{ad}$ and all non-invertibles objecs in $\mathcal{C}_{ad}$ are even dimensional, then they should be fixed by the non-trivial invertible object in $\mathcal{C}_{ad}$, by 5.5. Since the invertible objects are transparent in $\mathcal{C}_{ad}$, the non-trivial invertible object can not be a fermion, i.e. $\mathcal{Z}_{\mathcal{D}}(\mathcal{D}) \ncong$ sVec. If $\mathcal{Z}_{\mathcal{D}}(\mathcal{D}) \cong \operatorname{Rep}(\mathbb{Z}_2)$, we can de-equivariantize to find a $2m$-dimensional modular category. Thus $\mathcal{D}_{\mathbb{Z}_2}$ is pointed (by [15, Theorem 2.11 (i)]) and hence $\mathcal{D}$ is integral, an impossibility. □

Remark 5.8. It follows from Lemma 5.6 and Lemma 5.7 that if $\mathcal{C}$ is a strictly weakly integral prime modular category of dimension $16m$, with m odd square-free integer, then its universal grading group has order 4. Thus $\operatorname{FPdim} \mathcal{C}_{pt} = 4$.

We will focus on the self-dual case. Thus it suffices to consider strictly weakly integral self-dual categories of dimension $16m$ whose universal grading group has order 4.

Lemma 5.9. *Let $\mathcal{C}$ is a strictly weakly integral modular category of dimension $2^n m$, with $m > 1$ an odd square-free integer and $n \geq 4$. If $\dim \mathcal{C}_{pt} = 4$, then $\mathcal{Z}_{\mathcal{C}_{ad}}(\mathcal{C}_{ad}) = \mathcal{C}_{pt}$, and $\mathcal{C}_{pt}$ contains a boson.*

Proof. First note that $\dim \mathcal{C}_{ad} = 2^{n-2}m$. So the dimensions of the simple objects of $\mathcal{C}_{ad}$ are of the form 2^k, $k = 0, \ldots, \lfloor \frac{n}{2} \rfloor$, by Remark 5.4. Thus $2^{n-2}m = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{2k} a_k$. Since $n \geq 4$, then $4 \mid a_0$. Moreover $a_0 = 4$ because $\mathbf{1} \in \mathcal{C}_{ad}$. Thus $\mathcal{C}_{pt} \subset \mathcal{C}_{ad}$. On the other hand, $\mathcal{Z}_{\mathcal{C}_{ad}}(\mathcal{C}_{ad}) = \mathcal{Z}_{\mathcal{C}}(\mathcal{C}_{ad}) \cap \mathcal{C}_{ad} = \mathcal{C}_{pt} \cap \mathcal{C}_{ad} = \mathcal{C}_{pt}$.

Now consider X a non-invertible simple object in $\mathcal{C}_{ad}$ (which exists since $m > 1$). Recall that $2 \mid \mathrm{FPdim}\, Y$, for all non-invertible simple object $Y \in \mathcal{C}_{ad}$. Then, by Remark 5.5, we have that 2 divides the order of the group $G[X]$ of invertible objects that fixes X. Since $\mathbf{1} \in G[X]$, then $G[X] \geq 2$ Then, there exists $g \in G[X] \subseteq \mathcal{C}_{pt} = \mathcal{Z}_{\mathcal{C}_{ad}}(\mathcal{C}_{ad})$ of order 2. Since g fixes X and $g \in \mathcal{C}'_{ad} = \mathcal{C}_{pt}$ then g is not a fermion (see [29, Lemma 5.4]), and hence g must be a boson. $\square$

Lemma 5.10. *If m is an odd square-free integer and $\mathcal{C}$ is a strictly weakly integral modular category of dimension $16m$ with $\dim \mathcal{C}_{pt} = 4$ and GN-grading $\mathbb{Z}_2$, then the simple objects in the integral non-adjoint component are all 2-dimensional. Moreover, all the non-invertible simple objects in $\mathcal{C}_{ad}$ are also 2-dimensional.*

Proof. Let a_0, b_0, and c_0 denote the number of 1-dimensional, 2-dimensional, and 4-dimensional simples in $\mathcal{C}_{ad}$ respectively. Then $4m = \dim \mathcal{C}_{ad} = a_0 + 4b_0 + 16c_0$. In particular, $4 \mid a_0$. Since $\mathbf{1}$ is in $\mathcal{C}_{ad}$ then $a_0 > 0$. Thus $a_0 = 4$. Consequently, $\mathcal{C}_g$, the integral non-adjoint component must contain only 2-dimensional and 4-dimensional simple objects.

Moreover, $4m = 4b_1 + 16c_1$ where b_1 and c_1 are the number of 2- and 4-dimensionals in the integral non-adjoint component. Since m is odd we know that $b_1 \neq 0$, but $m = b_1 + 4c_1$, so b_1 is odd.

Since the GN-grading group and the universal grading group are not the same then $\mathcal{C}_{ad} \subsetneq \mathcal{C}_{int}$. So we have $\mathcal{Z}_{\mathcal{C}}(\mathcal{C}_{int}) \subsetneq \mathcal{Z}_{\mathcal{C}}(\mathcal{C}_{ad}) = \mathcal{C}_{pt}$. Notice that $\mathcal{C}_{int}$ is not modular, otherwise, by [30, Theorem 4.2], [12, Theorem 3.13], the category $\mathcal{C} \cong \mathcal{C}_{int} \boxtimes \langle g \rangle$ would be integral which contradicts the assumption of $\mathcal{C}$ being strictly weakly integral. Thus $\mathcal{Z}_{\mathcal{C}}(\mathcal{C}_{int}) = \langle g \rangle$ for some invertible g.

Wwe saw above that there is an odd number of 2-dimensionals in the non-adjoint integral component. Then, at least one these 2-dimensional objects are fixed by g. It follows from [29, Lemma 5.4] that g is a boson. Thus $\mathcal{C}_{int}$ is modularizable.

So $(\mathcal{C}_{int})_{\mathbb{Z}_2}$ is a $4m$-dimensional integral modular category and thus is pointed, by [5, Theorem 3.1]. In particular, $\mathcal{C}_{int}$ has character degrees 1 and 2. $\square$

Proposition 5.11. *If m is an odd square-free integer and $\mathcal{C}$ is a strictly weakly integral self-dual prime modular category of dimension $16m$ with GN-grading $\mathbb{Z}_2$, then $\mathcal{C}$ is spin modular.*

Proof. Suppose that the non-trivial invertibles, g, h, and gh, are all bosons.

In the same way as in the proof of Lemma 5.10 it can be shown that $\mathcal{Z}_{\mathcal{C}}(\mathcal{C}_{int}) = \langle g\rangle$, for some invertible g. Since the GN-grading group is $\mathbb{Z}_2$, the possible non-integral dimensions of simple objects are $\sqrt{t}$ and $2\sqrt{t}$, for some $t \in \mathbb{N}$.

Denote by $\mathcal{D}_a$, the subcategory $\mathcal{C}_{ad} \oplus \mathcal{C}_a$. Then $\mathcal{Z}_{\mathcal{C}}(\mathcal{D}) \subset \mathcal{Z}_{\mathcal{C}}(\mathcal{C}_{ad}) = \mathcal{C}_{pt}$. By primality and taking double centralizers we can conclude that $\mathcal{Z}_{\mathcal{C}}(\mathcal{D}) \neq \mathrm{Vec}, \mathcal{C}_{pt}$. In particular, there is a boson b such that $\mathcal{Z}_{\mathcal{C}}(\mathcal{D}) = \langle b\rangle$. Once again, by double centralizing, $b \neq g$. Similarly, denoting $\mathcal{D}_c$, the subcategory $\mathcal{C}_{ad} \oplus \mathcal{C}_c$, where $\mathcal{C}_c$ is the other non-integral component, we get $\mathcal{Z}_{\mathcal{C}}(\mathcal{D}) = \langle bg\rangle$. With out lost of generality, we can assume that $b = h$.

Since g does not centralize X we have $S_{g,X} \neq d_X$ and so by balancing and orthogonality of the S-matrix, g must move X. Similarly, g must move Y. On the other hand, g fixes the 2-dimensionals in $\mathcal{C}_{ad}$.

Now, assume there exist a simple $X \in \mathcal{D}_a$ of dimension $\sqrt{t}$ and a simple $Y \in \mathcal{D}_c$ of dimension $2\sqrt{t}$. Then $X \otimes Y$ is in the non-adjoint integral component of $\mathcal{C}$. So $X \otimes Y$ is a sum of 2-dimensional simple objects. But if Z is a 2-dimensional simple object such that $Z \otimes X \cong Y$ then $Y \cong Z \otimes Y \cong g \otimes Z \otimes X \cong g \otimes Y$. This is a contradiction since g does not fix Y.

Then, there is a fermion on $\mathcal{C}$. Moreover, both h and gh are fermions in $\mathcal{C}$.

□

Corollary 5.12. *Consider m an odd square-free integer and $\mathcal{C}$ a strictly weakly integral self-dual prime modular category of dimension $16m$ with GN-grading $\mathbb{Z}_2$. Let h and gh be the fermions in $\mathcal{C}$. Denote by $\mathcal{C}_a$ and $\mathcal{C}_b$ the non-integral components, then $\mathcal{Z}_{\mathcal{C}}(\mathcal{C}_{ad} \oplus \mathcal{C}_a) = \langle h\rangle$ and $\mathcal{Z}_{\mathcal{C}}(\mathcal{C}_{ad} \oplus \mathcal{C}_b) = \langle gh\rangle$.*

Moreover, there is no component containing only objects of dimension $2\sqrt{t}$ for some square-free integer t.

Proof. The proof of the first statement is contained in the proof of Proposition 5.11.

By [24, Theorem 3.10], the only possible dimensions of non-integral objects are $\sqrt{t}$ and $2\sqrt{t}$ for a square-free integer t.

Suppose to the contrary that there is some component containing only simples of dimension $2\sqrt{t}$. Without loss of generality we take this to be the $\mathcal{C}_a$ component. Then tensoring with h must permute these simples in a fixed point free manner. Then $4m = \mathrm{FPdim}\,\mathcal{C}_a = 4tk$, where k is the number of simples in $\mathcal{C}$. This is a contradiction since k is even and m odd. □

Lemma 5.13. *Let m be an odd square-free integer and $\mathcal{C}$ be a strictly weakly integral self-dual prime modular category of dimension $16m$ with GN-grading $\mathbb{Z}_2$. Then the non-integral objects have dimension $\sqrt{t}$ and $2\sqrt{t}$ for some* square-free even *integer t. Moreover each* *non-integral component contains an even number of objects of dimension $\sqrt{t}$ and an even number of objects of dimension $2\sqrt{t}$.*

Proof. By Corollary 5.12 we know that there is at least one object of dimension $\sqrt{t}$ in each non-integral component, let X and Y be such objects in $\mathcal{C}_a$ and $\mathcal{C}_b$ (under the notation of Corollary 5.12), respectively. Then $X \otimes Y$ must be in the integral non-adjoint component under the universal grading. In particular, by Lemma 5.10, $t = \dim X \otimes Y = 2s$ for some $s \in \mathbb{Z}$. The t is even. Furthermore, t is square-free since t divides m.

By Proposition 5.11, $\mathcal{C}$ is a spin modular category. Denote $\mathcal{D} = \mathcal{C}_{ad} \oplus \mathcal{C}_a$, then $\mathcal{Z}_{\mathcal{D}}(\mathcal{D}) \cong$ sVec and even multiplicity follows from the fact that the action of the fermion is fixed-point-free. The same holds for $\mathcal{C}_{ad} \oplus \mathcal{C}_b$. □

Corollary 5.14. *Let m be an odd square-free integer and $\mathcal{C}$ be a strictly weakly integral self-dual prime modular category of dimension $16m$ with GN-grading $\mathbb{Z}_2$. Then all of the non-integral simple objects have dimension $\sqrt{2m}$.*

Proof. By Proposition 5.11, $\mathcal{C}$ must be spin modular.

Suppose there are objects X and Y in the same component of dimension $2\sqrt{t}$ and $\sqrt{t}$ respectively. Then $X \otimes Y$ is in $\mathcal{C}_{ad}$. Since X and Y have different dimensions, no invertibles can appear as subobjects of $X \otimes Y$. Thus $X \otimes Y = \bigoplus N_{X,Y}^{Z_i} Z_i$ where Z_i are the 2-dimensional simple objects in $\mathcal{C}_{ad}$. But $N_{X,Y}^{Z_i} = N_{X,Z_i}^{Y}$. If we consider $Z_i \otimes X$ there are 3 possibilities: it is the direct sum of 2 simple objects of dimension $\sqrt{t}$, it is equal to a simple object of dimension $2\sqrt{t}$, or it equal to 2 copies of a simple object of dimension $\sqrt{t}$. If we tensor $Z \otimes X$ by the boson g, since Z is fixed by g, we get that $Z \otimes X = X_1 \oplus gX_1$, where X_1 is a $\sqrt{t}$-dimensional simple object in $\mathcal{C}_a$. This implies that $N_{X,Y}^{Z_i} = N_{X,Z_i}^{Y} = 0$ for all X, Y, and Z as above. But this is a contradiction because $X \otimes Y$ has dimension $2t$. It follows that the non-integral objects all have dimension $\sqrt{t}$ for some even square-free integer t, see Corollary 5.12.

Now suppose there exist non-integral simple objects $W \neq V$ from the same component such that $W \otimes V$ does not contain any invertible objects. Then the fermion that doesn't centralize W and V must fix them. On the other hand, this fermion must permute all elements of the adjoint subcategory since it is transparent in $\mathcal{C}_{ad}$. Hence $W \otimes V$ must decompose into an even number of 2-dimensional simples in the adjoint. Computing dimensions we see $4 \mid \dim W \otimes V = t$, which is impossible since t is square-free. Thus for any non-integral W, V in the same component, $W \otimes V$ contains an invertible, say a. Thus $a \otimes W = V$. In particular, this component can only contain two non-integral objects. The result now follows by equidimensionality of the universal grading. □

5.1. **General results and related questions.** Although we do not get as sharp a result in the case of $16m$ as we do in the cases $4m$ and $8m$, we can still give a significant amount of the structure under some assumptions:

Theorem 5.15. *Let $\mathcal{C}$ be a prime modular category of dimension $2^n m$ with m odd and square free.*

- *If $n = 1$, $\mathcal{C}$ is a pointed.*
- *[6, 4] If $n = 2$ or 3 $\mathcal{C}$ is a metaplectic modular category.*
- *If $n = 4$, $\mathcal{C}$ is self-dual and the GN grading is $\mathbb{Z}_2$ then $\mathcal{C}$ is either metaplectic or is obtained as a $\mathbb{Z}_2$ gauging of $\mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_{2m}, q)$ via an action of $\mathbb{Z}_2$ with exactly 2 fixed points.*

We do not know if the $\mathbb{Z}_2$ gauging of $\mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_{2m}, q)$ is self-dual or prime, so it is possible that a sharper result can be obtained. Moreover, we don't know what happens if the GN-grading is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Question 5.16. *How large can the GN-grading group be for a prime strictly weakly integral modular category?*

There is an upper bound, given in [11, Theorem 3.1], for weakly-integral modular categories of dimension $2^n m$, where $n \geq 0$ and m odd. In this case, the order of the GN-grading group is at most $2^{\frac{n}{2}}$.

It is of course straight forward to develop non-prime categories with large GN-grading via Deligne product of $\mathbb{Z}_2$-graded categories. A natural place to look for large GN-grading with fewer prime factors is through equivariantization. That is, perhaps one can find an equivariantization of one of these Deligne products that has a diverse prime factor. This of course leads to the following related question:

Question 5.17. *How do Deligne products behave under (de-)equivariantization?*

We have seen that for strictly weakly integral modular categories the structure can sometimes be determined by the dimension. Of course this is far from true for integral modular categories. There are many inequivalent modular categories of dimension 2^{2n}: simply take the twisted double of a finite group of order 2^n. On the other hand, the dimension of a non-integral category $\mathcal{C}$ can be factored over the Dedekind domain $\mathbb{Z}[\zeta_k]$ where k is the order of the T-matrix (see [7]) and it makes sense to look at such prime factorizations. For example:

Question 5.18. *How much of the structure of a non-integral modular category $\mathcal{C}$ can be determined from the primes dividing the ideal $\langle \dim \mathcal{C} \rangle$ (in an appropriate $\mathbb{Z}[\zeta_k]$)?*

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