1 Mathematical Reasoning $(Part I)^1$

Statements

DEFINITION 1. A statement is any declarative sentence or assertion that is either true or false.

A statement cannot be neither true nor false and it cannot be both true and false.

- 1. The integer 5 is odd.
- 2. The integer 24277151704311 is prime.
- 3. 15 + 7 = 22
- 4. Substitute the number 7 for x.
- 5. What is the derivative of $\cos x$?
- 6. Apple manufactures computers.
- 7. Apple manufactures the world's best computers.
- 8. Did you buy IBM?
- 9. I am telling a lie.
- 10. What happen when Pinocchio says: "My nose will grow now"?
- Set Terminology and Notation (very short introduction²) DEFINITIONS:

Set is well-defined collection of objects. **Elements** are objects or members of the set.

- Roster notation:
 - $A = \{a, b, c, d, e\}$ Read: Set A with elements a, b, c, d, e.
- Indicating a pattern:

 $B = \{a, b, c, ..., z\}$ Read: Set B with elements being the letters of the alphabet.

If a is an element of a set A, we write $a \in A$ that read "a belongs to A." However, if a does not belong to A, we write $a \notin A$.

¹This part is covered in Sections 1.1-1.3 in the textbook.

²We will study SETS in Chapter 2!

Very common sets:

- **R** is the set of all *real* numbers;
- $\mathbf{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, the set of all *integers*;
- $\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$, the set of all *positive integers*;
- $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$, the set of all *natural* numbers;

Other sets:

- **E** is the set of all *even integers*;
- **O** is the set of all *odd integers*;
- $n\mathbf{Z}$ is the set of all integers multiples of $n \ (n \in \mathbf{Z})$;

An **open sentence** is any declarative sentence containing one or more variables, each variable representing a value in some prescribing set, called the **domain** of the variable, and which becomes a statement when values from their respective domains are substituted for these variables.

- 1. P(x): x + 5 = 7
- 2. He is a student.

EXAMPLE 2. Discuss $P(x) : (x-3)^2 \le 1$ over **Z**.

EXAMPLE 3. Discuss $P(x, y) : x^2 + y^2 = 1$ when $x, y \in \mathbf{R}$.

The NEGATION of a Statement

DEFINITION 4. If P is a statement, then the negation of P, written $\neg P$ (read "not P"), is the statement "P is false".

Although $\neg P$ could always be expressed as

It is not that case that P.

there are usually better ways to express the statement $\neg P$.

- 1. P: The integer 77 is even.
- 2. $P: 5^3 = 120 \neg P:$ _____
- 3. P: The absolute value of the real number x is less than 5.

Compound Statements

Logical connectivity	write	read	meaning
Conjunction	$\mathbf{P} \wedge Q$	P and Q	Both P and Q are true
Disjunction	$\mathbf{P} \lor Q$	P or Q	P is true or Q is true

P: Ben is a student.

 $Q\!\!:$ Ben is a teaching assistant.

TRUTH TABLES

P	Q	$P \wedge Q$	$Q \wedge P$	$P \lor Q$	$Q \vee P$

EXAMPLE 5. Rewrite the following open sentences using disjunction or conjunction.

- (a) $P(x) : |x| \ge 10.$
- **(b)** P(x): |x| < 10.
- (c) $P(x): |4x+7| \ge 23.$

Implications

DEFINITION 6. Let P and Q be statements. The implication $P \Rightarrow Q$ (read "P implies Q") is the statement "If P is true, then Q is true."

EXAMPLE 7. If n is odd, then 3n + 7 is even.

The truth	ı table	for	imp	lication:
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P	Q	$P \Rightarrow Q$
Т	Т	Т
Т	F	F
F	Т	Т
F	F	Т

EXAMPLE 8. P: You earn an A on the final exam. Q: You get an A for your final grade.

 $P \Rightarrow Q$:

Different ways of expressing $P \Rightarrow Q$:

If P is true, then Q is true. Q is true if P is true. P implies Q. P is true only if Q is true. EXAMPLE 9. For a triangle T, let $P(T): T \text{ is equilateral} \qquad Q(T): T \text{ is isosceles.}$ State $P(T) \Rightarrow Q(T) \text{ in a variety of ways:}$

Necessary and Sufficient Conditions

 $P \Rightarrow Q$ also can be expressed as

$$P$$
 is sufficient for Q .
or
 Q is necessary for P .

Equivalently,

In order for Q to be true it is sufficient that P be true.

or

Q must be true in order to P to be true.

EXAMPLE 10. Consider the following open sentences P(x): x is a multiple of 4. Q(x): x is even. Complete:

- " \forall integer $x, P(x) \Rightarrow Q(x)$ " is _____.
- P(x) is a ______ condition for Q to be true.
- Q(x) is a _____ condition for P(x) to be true.
- Q(x) is not a ______ condition for P(x) to be true.

EXAMPLE 11. Consider the following open sentences

P(f): f is a differentiable function. Q(f): f is a continuous function.

Complete:

- " $\forall f, P(f) \Rightarrow Q(f)$ " is _____.
- " $\forall f, Q(f) \Rightarrow P(f)$ " is _____.

- Q(f) is a ______ condition for f to be differentiable, but not a ______ condition.
- P(f) is a _____ condition for f to be continuous.

REMARK 12. Note however, if $P \Rightarrow Q$ is true, then it is not necessary that P is true in order for Q to be true. Even if Q is true, P may be false.

Converse

DEFINITION 13. The statement $Q \Rightarrow P$ is called a **converse** of the statement $P \Rightarrow Q$.

EXAMPLE 14. If m and n are odd integers then m + n is even.

Rewrite the statement in symbols. Then write its converse both in symbols and words.

EXAMPLE 15. P: The function $f(x) = \sin x$ is differentiable everywhere. Q: The function $f(x) = \sin x$ is continuous everywhere.

$$P \Rightarrow Q \qquad \qquad Q \Rightarrow P$$

Biconditional "⇔"

For statements p and Q,

$$(P \Rightarrow Q) \land (Q \Rightarrow P)$$

is called the **biconditional** of P and Q and is denoted by $P \Leftrightarrow Q$. The biconditional $P \Leftrightarrow Q$ is stated as

"P is equivalent to Q." or "P if and only if Q." (or "P iff Q.")

or as "P is a necessary and sufficient condition for Q."

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
Т	Т			
Т	F			
F	Т			
F	F			

EXAMPLE 16. Let \vec{a} and \vec{b} be two non zero vectors. Then \vec{a} is orthogonal to \vec{b} iff $\vec{a} \cdot \vec{b} = 0$.

Tautologies and Contradictions

Tautology: statement that is always true Contradiction: statement that is always false

P	$\neg P$	$P \lor (\neg P)$	$P \land (\neg P)$
Т			
F			

Logical Equivalence

DEFINITION 17. Two compound statements are logically equivalent (write " \equiv ") if they have the same truth tables, which means they both are true or both are false.

Question: Are the statements $P \Rightarrow Q$ and $Q \Rightarrow P$ logically equivalent?

EXAMPLE 18. Let P and Q be statement forms. Determine whether the compound statements $\neg P \land Q$ and $\neg P \lor Q$ are logically equivalent (i.e. both true or both false).

P	Q		

REMARK 19. Let P and Q be statements. The biconditional $P \Leftrightarrow Q$ is a tautology if and only if P and Q are logically equivalent.

Some Fundamental Properties of Logical Equivalence

THEOREM 20. For the statement forms P, Q and R,

•
$$\neg(\neg P) \equiv$$

- Commutative Laws $P \lor Q \equiv$ $P \land Q \equiv$
- Associative Laws $P \lor (Q \lor R) \equiv$ $P \land (Q \land R) \equiv$
- Distributive Laws $P \lor (Q \land R) \equiv$ $P \land (Q \lor R) \equiv$
- De Morgan's Laws $\neg (P \lor Q) \equiv (\neg P) \land (\neg Q)$ $\neg (P \land Q) \equiv (\neg P) \lor (\neg Q)$

Proof. Each part of the theorem is verified by means of a truth table.

P	Q			

THEOREM 21. For statements P and Q,

$$\neg (P \Rightarrow Q) \equiv P \land (\neg Q).$$

Proof.

Quantified Statements and Negations

EXAMPLE 22. Consider the following open sentence: $P(n): \frac{2n^2 + 5 + (-1)^n}{2} \text{ is prime.}$ How to convert this open sentence into a statement?

An open sentence can be made into a statement by using quantifiers. **Universal**: $\forall x$ means for all/for every assigned value a of x. **Existential**: $\exists x$ means that for some assigned values a of x. Quantified statements

in symbols	in words	
$\forall x \in D, P(x).$	For every $x \in D$, $P(x)$.	
	If $x \in D$, then $P(x)$.	
$\exists x \in D \ni P(x)$	There exists x such that $P(x)$.	

Once a quantifier is applied to a variable, then the variable is called a **bound** variable. The variable that is not bound is called a **free** variable.

- 1. The area of a rectangle is its length times its width. Quantifiers:
- 2. A triangle may be equilateral. Quantifiers:
- 3. 15 5 = 10Quantifiers:
- 4. A real-valued function that is continuous at 0 is not necessarily differentiable at 0. Quantifiers:

EXAMPLE 23. Rewrite the following statements in symbols using quantifiers. Introduce variables, where appropriate.

- a) For every real number x, x + 5 = 7.
- **b**) All positive real numbers have a square root.

c) The sum of an even integer and an odd integer is even.

d) For every integer n, either $n \leq 1$ or $n^2 \geq 4$.

NEGATIONS

- 1. All continuous functions are differentiable.
- 2. P: Every car on the parking lot #47 was with valid permit.
 - $\neg P$ _____
- 3. P : There exist real numbers a and b such that $(a + b)^2 = a^2 + b^2$.
 - $\neg P$ _____

Rules to negate statements with quantifiers:

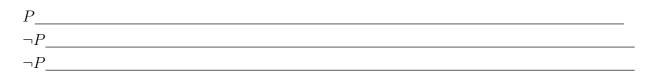
 $\neg(\forall x \in D, P(x)) \equiv$ $\neg(\exists x \in D \ni P(x)) \equiv$ $\neg(\forall x \in D, (P(x) \lor Q(x)) \equiv$ $\neg(\forall x \in D, (P(x) \land Q(x)) \equiv$ $\neg(\exists x \in D \ni (P(x) \lor Q(x)) \equiv$ $\neg(\exists x \in D \ni (P(x) \land Q(x)) \equiv$

EXAMPLE 24. Negate the statements below using the following steps:

- 1. Rewrite P in symbols using quantifiers.
- 2. Express the negation of P in symbols using the above rules.
- 3. Express $\neg P$ in words.
- **a)** P: If n is an odd integer then 3n + 7 is odd.



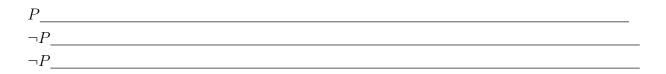
b) P: There exists a positive integer n such that m(n+5) < 1 for every integer m.



c) P: If n is an integer and n^2 is a multiple of 4 then n is a multiple of 4.



d) P: For every even integer n there exists an integer m such that n = 2m.



e) P: There exists a prime number p which is greater then 7 and less than 10.