1 Mathematical Reasoning (Part II)[1]

Proving Statements Containing Implications.

Properties of Integers:

FACT 1 *The negative of every integer is an integer.*

FACT 2 *The sum (and difference) of every two integers is an integer.*

FACT 3 *The product of every two integers is an integer.*

FACT 4 *Every integer is either even, or odd.*

- **DIRECT PROOF**
  - Assume that P is true.
  - Draw out consequences of P.
  - Use these consequences to prove Q is true.

EXAMPLE 1. *If n is an even integer, then $5n^5$ is an even integer.*

EXAMPLE 2. Evaluate the proposed proof of the following result:

If a is an even integer and b is an odd integer, then $3a-5b$ is odd.

Proof. Let a be an even integer and b be an odd integer. Then $a = 2n$ and $b = 2n+1$ for some integer n. Therefore,

$$3a-5b = 3(2n)-5(2n+1) = 6n-10n-5 = -4n-5 = 2(-2n-2)-1.$$

Since $-2n-2$ is an integer, $3a-5b$ is odd. $\square$

[1]This part is covered in Sections 1.3-1.4 in the textbook.

EXAMPLE 3. *The sum of every two odd integers is even.*

EXAMPLE 4. *Let x be an integer. If $5x-7$ is odd, then $9x+2$ is even.*

- **PROOF BY CASES**

  EXAMPLE 5. *If n is an integer, then $n^2 + 3n + 4$ is an even integer.*

  Hint: Use the following fact: "*Every integer number is either even. or odd.*"

`Negating an implication:` **Counterexamples**

Recall that (see Theorem 21 in the previous part)

$$\neg(P \Rightarrow Q) \equiv P \wedge (\neg Q)$$

EXAMPLE 6. *P: If n is an integer and n^2 is a multiple of 4 then n is a multiple of 4.*
Question: Is the following "proof" valid?
Let $n = 6$. Then $n^2 = 6^2 = 36$ and 36 is a multiple of 4, but 6 is not a multiple of 4. Therefore, the statement P is FALSE.□

REMARK. The negation of an implication is not an implication!

EXAMPLE 7. *Negate the statement: "For all $x, P(x) \Rightarrow Q(x)$."*

The value assigned to the variable x that makes $P(x)$ true and $Q(x)$ false is called a **counterexample** to the statement "For all $x, P(x) \Rightarrow Q(x)$."

EXAMPLE 8. Disprove the following statement:

If a real-valued function is continuous at some point, then this function is differentiable there.

Contrapositive

DEFINITION 9. *The statement* $\neg Q \Rightarrow \neg P$ *is called the* **contrapositive** *of the statement* $P \Rightarrow Q$.

EXAMPLE 10. *Let* P *and* Q *be statement forms. Prove that* $\neg Q \Rightarrow \neg P$ *is logically equivalent to* $P \Rightarrow Q$.

P	Q	$P \Rightarrow Q$	$\neg Q$	$\neg P$	$\neg Q \Rightarrow \neg P$

Methods to prove an implication $P \Rightarrow Q$ (continued)

- **CONTRAPOSITIVE PROOF** (based on the equivalence $(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)$)
  - Assume that $\neg Q$ is true.
  - Draw out consequences of $\neg Q$.
  - Use these consequences to prove $\neg P$ is true.
  - It follows that $P \Rightarrow Q$.

  REMARK 11. If you use a contrapositive method, you must declare it in the beginning and then state **what is sufficient to prove.**

EXAMPLE 12. *Let* x *be an integer. If* $5x - 7$ *is even, then* x *is odd.*

THEOREM 13. *Let n be an integer. Then n is even if and only if n^2 is even.*

Proof.

REMARK 14. $(P \Leftrightarrow Q) \equiv (\neg P \Leftrightarrow \neg Q)$

COROLLARY 15. *Let n be an integer. Then n is odd iff n^2 is odd.*

Methods to prove an implication $P \Rightarrow Q$ (continued)

EXAMPLE 16. *Let S and C be statement forms. Prove that $\neg S \Rightarrow (C \wedge \neg C)$ is logically equivalent to S.*

- **PROOF BY CONTRADICTION**
  - Assume that P is true.
  - To derive a contradiction, assume that $\neg Q$ is true.
  - Prove a false statement C, using negation $\neg(P \Rightarrow Q) \equiv (P \wedge \neg Q)$.
  - Prove $\neg C$. It follows that Q is true. (The statement $C \wedge \neg C$ must be false, i.e. a contradiction.)

PROPOSITION 17. *If m and n are integers, then $m^2 - 4n \neq 2$.*

Proof.

PROPOSITION 18. *Let a, b, and c be integers. If $a^2 + b^2 = c^2$ then a or b is an even integer.*

Proof.

DEFINITION 19. *A real number x is* **rational** *if $x = \frac{m}{n}$ for some integer numbers m and n. Also, x is* **irrational** *if it is not rational, that is*

PROPOSITION 20. *The number $\sqrt{2}$ is irrational.*