2. Sets

2.1&2.2: Sets and Subsets. Combining Sets.

• **Set Terminology and Notation**

DEFINITIONS:

Set is a well-defined collection of objects.

Elements are objects or members of the set.

• **Roster notation:**

$A = \{a, b, c, d, e\}$ Read: Set A with elements a, b, c, d, e.

• **Indicating a pattern:**

$B = \{a, b, c, ..., z\}$ Read: Set B with elements being the letters of the alphabet.

Set-builder notation:

$S = \{x|p(x)\}$ (This means that S consists of all those elements x satisfying some condition $p(x)$ concerning x.)

$B = \{x|x$ is a letter of the English alphabet$\}$

$C = \{x|x$ is a student in this classroom$\}$

The symbol "|" is read "such that".

If a is an element of a set A, we write $a \in A$ that read "a belongs to A." However, if a does not belong to A, we write $a \notin A$.

Very common sets:

• **R** is the set of all *real* numbers;

• **Z** $= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, the set of all *integers*;

• **Z**$^+ = \{1, 2, 3, \ldots\}$, the set of all *positive integers*;

• **N** $= \{0, 1, 2, 3, \ldots\}$, the set of all *natural* numbers;

• **Q** is the set of all *rational numbers*;

Other sets:

• **E** is the set of all *even integers*;

• **O** is the set of all *odd integers*;

• n**Z** is the set of all integers multiples of n ($n \in$ **Z**);

Application to set definitions

Set-builder notation:

$A = \{x \in S | P(x)\}$ is the set of all elements x in S such that the open sentence $P(x)$ is a true statement.

EXAMPLE 1. *Use the set-builder notation to describe the following sets:*

a) N

b) Q

c) O

d) E

e) 5Z

Interval notation:

EXAMPLE 2. **a)** $\{n \in \mathbf{Z} | \ 3 \leq n < 10\} =$

b) $\{x \in \mathbf{R} | \ (x \geq 0) \wedge (x \in \mathbf{Z})\} =$

c) $\{x \in \mathbf{R} | \ -2015 \leq x \leq 2016\} =$

d) $\{x \in \mathbf{R} | \ |x + 5| \leq 7\} =$

Intervals:

- bounded intervals:

1. closed interval $[a, b] =$

2. open interval $(a, b) =$

3. half-open,half-closed interval $(a, b] =$

4. half-closed,half-open interval $[a, b) =$

   - unbounded intervals:

5. $[a, \infty) =$

6. $(a, \infty) =$

7. $(-\infty, a] =$

8. $(-\infty, a) =$

9. $(-\infty, \infty) =$

infinite set

finite set

cardinality of A, $|A|$

Subsets

Two sets, A and B, are **equal**, written $A = B$, if and only if they have exactly the same elements. (NOTE: they do not have to be in the same order!). If every element in set A is also an element in set B, then A is a subset of B, written $A \subseteq B$. If $A \subseteq B$, but $A \neq B$, then A is a **proper** subset of B, written $A \subset B$.

EXAMPLE 3. *Given*
$A = \{a, e, i, o, u\}$
$B = \{u, i, e, a, o\}$
$C = \{a, e, i, o\}$. *Then*
$D = \{e, i, o, a\}$

EXAMPLE 4. *Which of the following are TRUE?*

1. $\mathbf{Z}^+ \subset \mathbf{Z}$

2. $\mathbf{Z}^+ \subseteq \mathbf{Z}$

3. $\mathbf{N} \subseteq \mathbf{Z}^+$

4. $\mathbf{Z} \subset \mathbf{Q} \subseteq \mathbf{R}$

The **empty set** is the set that doesn't have any elements, denoted by $\emptyset$ or $\{\}$. The **universal set** is the set that contains all of the elements for a problem, denoted by U.

EXAMPLE 5. *Give all the subsets for these sets.*
   (a) $A = \{0, 1\}$ (b)$X = \{a, b, y\}$

EXAMPLE 6. *List all elements of the following set:*

$$\{x \in \mathbf{R}| \ \sin x = 2\}$$

EXAMPLE 7. *Given* $A = \{0, 1, 2, \ldots, 8\}$, $B = \{1, 3, 5, 7\}$, $C = \{3, 5, 1, 7\}$, *and* $D = \{5, 3, 1\}$, $E = \emptyset$ *then which of the following are TRUE?*

(**a**)$B = C$ (**b**)$B \subseteq C$ (**c**)$B \subset C$ (**d**)$C \subseteq B$ (**e**)$D \subset B$

(**f**)$D \subseteq B$ (**g**)$B \subset D$ (**h**)$8 \in A$ (**i**) $\{4, 6\} \subset A$ (**j**)$1, 5 \subset A$

(**k**)$9 \notin C$ (**l**)$D \subseteq D$ (**m**)$\emptyset = 0$ (**n**)$0 \in E$ (**o**)$A \in A$

VENN DIAGRAMS - a visual representation of sets (the universal set U is represented by a

rectangle, and subsets of U are represented by regions lying inside the rectangle).

EXAMPLE 8. *Use Venn diagrams to illustrate the following statements:*

(**a**) $A = B$

(**b**) $A \subset B \subset C$

(**c**) *A and B are not subsets of each other.*

• **OPERATIONS OF SETS**

Let A and B be sets. The **union** of A and B, written $A \cup B$, is the set of all elements that belong to either A or B or both. Symbolically:

$$A \cup B = \{x | x \in A \vee x \in B\}.$$

Let A and B be sets. The **intersection** of A and B, written $A \cap B$, is the set of all elements in common with A and B. Symbolically:

$$A \cap B = \{x | x \in A \ \wedge \ x \in B\}.$$

U

DEFINITION 9. *Let A and B be sets. The* **complement of A in B** *denoted $B - A$, is* $\{b \in B | b \notin A\}$.

REMARK 10. For convenience, if U is a universal set and A is a subset in U, we will write $U - A = \bar{A}$, called simply the **complement** of A.
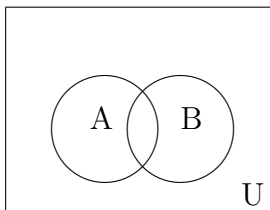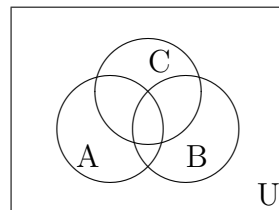
U

EXAMPLE 11. *Find $\bar{U}$ and $\bar{\emptyset}$.*

set notation	$=$	$\subset, \subseteq$	$\cup$	$\cap$	$\bar{\square}$	$\emptyset$
logical connectivity						

EXAMPLE 12. *Shade the Venn diagrams below to represent the following sets*

(a) $\bar{A} \cap B$

(b) $\overline{(A \cup B)} \cap C$

EXAMPLE 13. *Use these sets to find the following:* $U = \{0, 1, 2, ..., 9, 10\}$, $A = \{0, 2, 4, 6, 8, 10\}$, $B = \{1, 3, 5, 7, 9\}$, $C = \{0, 3, 4, 5, 7\}$, $D = \{0, 6, 7, 9\}$, $E = \{1, 3, 8, 9\}$

(a) $B \cup D$

(b) $A \cup B$

(c) $\bar{C}$

(d) $D - E$

(e) $E - D$

EXAMPLE 14. *Use set notation to reformulate the following theorem: "Every real-valued continuous function on $[a, b]$ is integrable on $[a, b]$." Also describe a universal set. Discuss the converse statement.*

THEOREM 15. *Let A and B be sets contained in some universal set U. Then $A \subseteq B$ if and only if $\bar{B} \subseteq \bar{A}$.*

Proving set properties

Use the following tautologies:

- $x \in A \cap B \Leftrightarrow (x \in A \wedge x \in B)$

- $x \in A \cup B \Leftrightarrow$

- $x \in A - B \Leftrightarrow$

- $A = B \Leftrightarrow (x \in A \Leftrightarrow x \in B)$

- $A \subseteq B \Leftrightarrow (x \in A \Rightarrow x \in B)$

Question: Let $A = \{n \in \mathbf{Z} | n \text{ is even}\}$ and $B = \{n \in \mathbf{Z} | n^2 \text{ is even}\}$. Are these sets the same?

Methods:

- To prove $A \subseteq B$ it is sufficient to prove $x \in A \Rightarrow x \in B$.

- To prove $A = B$ it is sufficient to prove $x \in A \Leftrightarrow x \in B$.

- To prove $A = B$ it is sufficient to prove $A \subseteq B$ and $B \subseteq A$.

- To show that $A = \emptyset$ it is sufficient to show that $x \in A$ implies a false statement.

EXAMPLE 16. *Let $A = \{n \in \mathbb{Z} | n = 3t - 2 \text{ for some } t \in \mathbb{Z}\}$ and $B = \{n \in \mathbb{Z} | n = 3t + 1 \text{ for some } t \in \mathbb{Z}\}$. Prove that $A = B$.*

EXAMPLE 17. *Let A and B be sets. Show that $(A - B) \cap B = \emptyset$.*

PROPOSITION 18. *Let $A, B,$ and C be sets, and suppose $A \subseteq B$ and $B \subseteq C$. Then $A \subseteq C$.*

Fundamental properties of sets

THEOREM 19. *The following statements are true for all sets A, B, and C.*

1. $A \cup B = B \cup A$ (commutative)

2. $A \cap B = B \cap A$ (commutative)

3. $(A \cup B) \cup C = A \cup (B \cup C)$ (associative)

4. $(A \cap B) \cap C = A \cap (B \cap C)$ (associative)

5. $A \subseteq A \cup B$.

6. $A \cap B \subseteq A$.

7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (distributive)

8. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (distributive)

9. *The empty set is a subset of every set. (Namely, for every set A, $\emptyset \subseteq A$. If $A \neq \emptyset$, then $\emptyset \subset A$.).*

10. $A \cup \emptyset = A$.

11. $A \cap \emptyset = \emptyset$.

    DeMorgan's Laws: *If A and B are the sets contained in some universal set U then*

12. $\overline{A \cup B} = \bar{A} \cap \bar{B}$.

13. $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

EXAMPLE 20. *Prove the following statement. Let A and B be subsets of a universal set U. Then $A \subseteq B \Leftrightarrow A \cup B = B$.*

Cartesian Product

DEFINITION 21. *Let A and B be sets. The **Cartesian product** of A and B, written $A \times B$, is the following set:*

$$A \times B = \{(a, b) | \ a \in A \wedge b \in B\}.$$

Informally, $A \times B$ is the set of **ordered** pairs of objects.

EXAMPLE 22. *Given $A = \{0, 1\}$ and $B = \{4, 5, 6\}$.*

(a) *Does the pair $(6, 1)$ belong to $A \times B$?*

(b) *List the elements of $A \times B$.*

(c) *What is the cardinality of $A \times B$?*

(d) *List the elements of $A \times A \times A$.*

(e) *Does the triple* $(1, 6, 4)$ *belong to* $A \times B \times B$?

(f) *Describe the following sets* $R \times R$, $R \times R \times R$.

EXAMPLE 23. *Let* A, B, C *and* D *be sets. If* $A \subseteq C$ *and* $B \subseteq D$, *then* $A \times B \subseteq C \times D$.

EXAMPLE 24. *For the sets* A, B *and* C *prove that*

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

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2.3 Collections of Sets

• **Power set**

DEFINITION 25. *Let* A *be a set. The power set of* A, *written* $P(A)$, *is*

$$P(A) = \{X \mid X \subseteq A\}.$$

EXAMPLE 26. *Let $A = \{-1, 0, 1\}$.*

1. *Write all subsets of A.*

2. *Find all elements of power set of A.*

3. *Find $|P(A)|$.*

4. *Write 3 subsets of $P(A)$.*

5. *Find $|P(P(A))|$.*

6. *Find $P(\emptyset)$, $P(P(\emptyset))$ and $P(\{-1\})$.*

• **Indexed Sets**

DEFINITION 27. *Let I be a set. An indexed collection of set $\{A_\alpha\}_{\alpha \in I}$ represents a collection of sets such that for every $\alpha \in I$, there is a corresponding set A_α. In this case we call I the* **indexed set.**

Collection of sets	Indexed set	
$A_0, A_1, A_2, A_3, \ldots, A_{2016}$		
B_3, B_6, B_9, B_{77}		
$C_5, C_{10}, C_{15}, \ldots, C_{2015}$		

- **Union and Intersection**

EXAMPLE 28. *Negate the following statement forms.*

(a) $x \in \bigcup_{\alpha \in I} A_\alpha \Leftrightarrow \exists \alpha \in I \ni x \in A_\alpha$

(b) $x \in \bigcap_{\alpha \in I} A_\alpha \Leftrightarrow \forall \alpha \in I, x \in A_\alpha$

EXAMPLE 29. *Given $B_i = \{i, i+1\}$ for $i = 1, 2, \ldots, 10$. Determine the following*

(a) $\bigcap_{i=1}^{10} B_i$

(b) $B_i \cap B_{i+1}$

(c) $\bigcap_{i=k}^{k+1} B_i$ *where $1 \le k < 10$.*

(d) $\bigcap_{i=j}^{k} B_i$ *where $1 \le j < k \le 10$.*

EXAMPLE 30. $A_n = \left\{x \in \mathbf{R}|\ -\frac{1}{n} \leq x \leq \frac{1}{n}\right\},\quad n \in \mathbf{Z}^+.$ *Find* $\displaystyle\bigcup_{n \in \mathbf{Z}^+} A_n$ *and* $\displaystyle\bigcap_{n \in \mathbf{Z}^+} A_n.$

● **Partitions of sets**

Disjoint sets   *Mutually exclusive*

DEFINITION 31. *A partition* $\mathcal{P}$ *of* A *is a subset of the power set* $P(A)$ *such that*

1. $X \in \mathcal{P} \Rightarrow X \neq \emptyset$

2. $\displaystyle\bigcup_{X \in \mathcal{P}} X = A$

3. *If* $X, Y \in \mathcal{P}$ *and* $X \neq Y$ *then* $X \bigcap Y = \emptyset.$

EXAMPLE 32. *Determine which of these sets are partitions of* $A = \{1, 2, 3, 4, 5, 6\}$:

$$\mathcal{P}_1 = \{\{1, 3, 6\}, \{2, 4\}, \{5\}\}$$

$$\mathcal{P}_2 = \{\{1, 2, 3\}, \{4\}, \emptyset, \{5, 6\}\}$$

$$\mathcal{P}_3 = \{\{1, 2\}, \{3, 4, 5\}, \{5, 6\}\}$$

$$\mathcal{P}_4 = \{\{1, 4\}, \{3, 5\}, \{2\}\}$$

EXAMPLE 33. *Let $A = 2\mathbf{Z}$ and $B = \{n \in \mathbf{Z} | n = 2t + 1$ for some $t \in \mathbf{Z}\}$. Show that $\{A, B\}$ is a partition of $\mathbf{Z}$.*

EXAMPLE 34. *Give an example of a partition for $A = \{1, 2, \ldots, 12\}$ such that $|\mathcal{P}| = 5$.*

EXAMPLE 35. *Give an example of a partition for $\mathbf{R}$.*