

3 FUNCTIONS

3.1 Definition and Basic Properties

DEFINITION 1. Let A and B be nonempty sets. A **function** f from A to B is a rule that assigns to each element in the set A one and only one element in the set B .

We call A the **domain** of f and B the **codomain** of f .

We write $f : A \rightarrow B$ and for each $a \in A$ we write $f(a) = b$ if b is assigned to a .

Using diagram

DEFINITION 2. Two functions f and g are **equal** if they have the same domain and the same codomain and if $f(a) = g(a)$ for all a in domain.

EXAMPLE 3. Let $A = \{2, 4, 6, 10\}$ and $B = \{0, 1, -1, 8\}$. Write out three functions with domain A and codomain B .

Some common functions

- *Identity* function $i_A : A \rightarrow A$ maps every element to itself:
- *Linear* function $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by
- *Constant* function $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

DEFINITION 4. Let A and B be nonempty sets. We define

$$F(A, B) =$$

the set of all functions from A to B .

If $A = B$, we simply write $F(A)$.

Image of a Function

DEFINITION 5. Let $f \in F(A, B)$. The **image** of f is

$$\text{Im}(f) = \{y \in B \mid y = f(x) \text{ for some } x \in A\}.$$

EXAMPLE 6. Let $f \in F(\mathbb{R})$ and $g \in F(\mathbb{R}, [0, \infty))$ defined by $f(x) = g(x) = x^4$. Find $\text{Im}(f)$ and $\text{Im}(g)$.

EXAMPLE 7. Let $f, g : F(\mathbb{R})$ be defined by $f(x) = \cos x$ and $g(x) = |\cos x|$. Find $\text{Im}(f)$ and $\text{Im}(g)$.

Image of a Set

DEFINITION 8. Let $f : A \rightarrow B$ be a function. If $X \subseteq A$, we define $f(X)$, the **image** of X under f , by

$$f(X) = \{y \in B \mid y = f(x) \text{ for some } x \in X\}.$$

EXAMPLE 9. Let $f \in F(A, B)$. Complete:

(a) If $X \subseteq A$ then $f(X) \subseteq \underline{\hspace{1cm}} \subseteq \underline{\hspace{1cm}}$

(b) $f(A) = \underline{\hspace{1cm}}$

EXAMPLE 10. $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = \cos x$. Find $f([- \pi/2, \pi/2])$.

Inverse Image

DEFINITION 11. Let $f : A \rightarrow B$ be a function and let W be a subset of its codomain (i.e. $W \subseteq B$). Then the **inverse image** of W (written $f^{-1}(W)$) is the set

$$f^{-1}(W) = \{a \in A \mid f(a) \in W\}.$$

EXAMPLE 12. Let $f \in F(\mathbb{R})$ be defined by $f(x) = x^4$. Find the following sets:

(a) $f([1, 2]) =$

(b) $f([-2, -1]) =$

(c) $f^{-1}([1, 16]) =$

(d) $f^{-1}([-16, -1]) =$

(e) $f^{-1}([-1, 1]) =$

(f) $f((-\infty, \infty)) =$

(g) $f^{-1}(\mathbb{R}) =$

(h) $f^{-1}(\mathbb{R}^+) =$

EXAMPLE 13. Let $A = \{a, b, c, d, e, f\}$ and $B = \{7, 9, 11, 12, 13\}$ and let the function $g : A \rightarrow B$ be given by

$$g(a) = 11, g(b) = 9, g(c) = 9, g(d) = 11, g(e) = 9, g(f) = 7.$$

Find

$$f^{-1}(\{7, 9\}) =$$

$$f^{-1}(\{12, 13\}) =$$

$$f^{-1}(\{11, 12\}) =$$

Summary

Let $f : A \rightarrow B$. The above definitions imply the following tautologies

- $(y \in \text{Im}(f)) \Leftrightarrow (\exists x \in A \ni f(x) = y)$.
- $(y \in f(X)) \Leftrightarrow (\exists x \in X \ni f(x) = y)$.
- $(x \in f^{-1}(W)) \Leftrightarrow (f(x) \in W)$.
- If $W \subseteq \text{Im}(f)$ then $(S = f^{-1}(W)) \Rightarrow (f(S) = W)$.

EXAMPLE 14. Let $S = \{y \in \mathbb{R} | y \geq 0\}$. Prove that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = x^4$ then $\text{Im}(f) = S$.

EXAMPLE 15. $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = 5x - 4$. Find $f([0, 1])$. Justify your answer.

EXAMPLE 16. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = 3x + 4$ and let $W = \{x \in \mathbb{R} | x > 0\}$. Find $f^{-1}(W)$.

EXAMPLE 17. Prove that $f(\mathbb{E}) = \mathbb{O}$, where $f \in F(\mathbb{Z})$ is defined by $f(n) = \begin{cases} n - 1, & \text{if } n \in \mathbb{E}, \\ n + 1, & \text{if } n \in \mathbb{O}. \end{cases}$

EXAMPLE 18. Let $f \in F(\mathbb{Z})$ be defined by $f(n) = \begin{cases} n/2 & \text{if } n \in \mathbb{E}, \\ n+1 & \text{if } n \in \mathbb{O}. \end{cases}$ Compute the following.

(a) $f^{-1}(\{6, 7\}) =$

(b) $f^{-1}(\mathbb{O})$

PROPOSITION 19. Let $f \in F(A, B)$. If $X \subseteq Y \subseteq A$ then $f(X) \subseteq f(Y)$.

Proof.

PROPOSITION 20. Let $f \in F(A, B)$. If $X \subseteq A$ and $Y \subseteq A$ then

(a) $f(X \cup Y) = f(X) \cup f(Y)$.

(b) $f(X \cap Y) \subseteq f(X) \cap f(Y)$.

Proof

PROPOSITION 21. Let $f \in F(A, B)$. If W and V are subsets of B then

(a) $f^{-1}(W \cup V) = f^{-1}(W) \cup f^{-1}(V)$.

(b) $f^{-1}(W \cap V) = f^{-1}(W) \cap f^{-1}(V)$.

Section 3.2 Surjective and Injective Functions

Surjective functions (“onto”)

DEFINITION 22. Let $f : A \rightarrow B$ be a function. Then f is **surjective** (or a surjection) if the image of f coincides with its codomain, i.e.

$$\text{Im}f = B.$$

Note: surjection is also called ‘‘onto’’.

Proving surjection:

We know that for all $f : A \rightarrow B$: _____

Thus, to show that $f : A \rightarrow B$ is a surjection it is sufficient to prove that _____

In other words,

to prove that $f : A \rightarrow B$ is a surjective function it is sufficient to show that _____

Question: How to disprove surjectivity?

EXAMPLE 23. Determine which of the following functions are surjective.

(a) Identity function

(b) $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^4$.

(c) $g : \mathbb{R} \rightarrow \mathbb{R}^+ \cup \{0\}, g(x) = x^4$.

$$(d) f : \mathbb{Z} \rightarrow \mathbb{Z}, f(n) = \begin{cases} n - 2 & \text{if } n \in \mathbb{E}, \\ 2n - 1 & \text{if } n \in \mathbb{O}. \end{cases}$$

$$(e) f : \mathbb{Z} \rightarrow \mathbb{Z}, f(n) = \begin{cases} n + 1 & \text{if } n \in \mathbb{E}, \\ n - 3 & \text{if } n \in \mathbb{O}. \end{cases}$$

Injective functions (“one to one”)

DEFINITION 24. Let $f : A \rightarrow B$ be a function. Then f is **injective** (or an *injection*) if whenever $a_1, a_2 \in A$ and $a_1 \neq a_2$, we have $f(a_1) \neq f(a_2)$.

Recall that surjection is also called “onto”. Using diagram:

EXAMPLE 25. Given $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$.

(a) Write out an injective function with domain A and codomain B . Justify your answer.

(b) Write out a non injective function with domain A and codomain B . Justify your answer.

Proving injection:

Let $P(a_1, a_2) : a_1 \neq a_2$ and $Q(a_1, a_2) : f(a_1) \neq f(a_2)$.

Then by definition f is injective if _____.

Using contrapositive, we have _____.

In other words, to prove injection show that:

Question: How to disprove injectivity?

EXAMPLE 26. Prove or disprove injectivity of the following functions.

(a) $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = \sqrt[5]{x}$.

(b) $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^4$.

(c) $f : \mathbb{Z} \rightarrow \mathbb{Z}, f(n) = \begin{cases} n/2 & \text{if } n \in \mathbb{E}, \\ 2n & \text{if } n \in \mathbb{O}. \end{cases}$

Discussion Exercise.

- Must a strictly increasing or decreasing function be injective?

- Must an injective function be strictly increasing or decreasing?

EXAMPLE 27. Prove or disprove injectivity of the following functions. In each case, $f \in F(\mathbb{R})$.

(a) $f(x) = 3x^5 + 5x^3 + 2x + 2016$.

(b) $f(x) = x^{12} + x^8 - x^4 + 11$.

Bijjective functions

DEFINITION 28. A function that is both surjective and injective is called **bijjective** (or bijection.)

EXAMPLE 29. Determine which of the following functions are bijjective.

(a) $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^3$.

(b) $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$.

EXAMPLE 30. Prove that $f \in F(\mathbb{R} - \{2\}, \mathbb{R} - \{3\})$ defined by $f(x) = \frac{3x}{x-2}$ is bijjective.

3.3 Composition and Invertible Functions

Composition of Functions

DEFINITION 31. Let A , B , and C be nonempty sets, and let $f \in F(A, B)$, $g \in F(B, C)$. We define a function

$$gf \in F(A, C),$$

called the **composition** of f and g , by

$$gf(a) =$$

Alternative notation for composition of f and g : $g \circ f$

EXAMPLE 32. Let $f, g \in F(\mathbf{R})$ be defined by $f(x) = e^x$ and $g(x) = x \sin x$. Find fg and gf .

EXAMPLE 33. Let $f \in F(\mathbf{R} - \{0, 1\})$ be defined by $f(x) = 1 - \frac{1}{x}$. Determine $f \circ f \circ f$.

EXAMPLE 34. Let $f, g \in F(\mathbf{Z})$ be defined by

$$f(n) = \begin{cases} n + 4, & \text{if } n \in \mathbf{E} \\ 2n - 3, & \text{if } n \in \mathbf{O} \end{cases} \quad g(n) = \begin{cases} 2n - 4, & \text{if } n \in \mathbf{E} \\ (n - 1)/2, & \text{if } n \in \mathbf{O} \end{cases}$$

Find gf and fg .

PROPOSITION 35. *Let $f \in F(A, B)$ and $g \in F(B, C)$. Then*

i. *If f and g are surjections, then gf is also a surjection.*

Proof.

ii. *If f and g are injections, then gf is also an injection.*

Proof.

COROLLARY 36. *If f and g are bijections, then gf is also a bijection.*

PROPOSITION 37. Let $f \in F(A, B)$. Then $fi_A = f$ and $i_Bf = f$.

Inverse Functions

DEFINITION 38. Let $f \in F(A, B)$. Then f is **invertible** if there is a function $f^{-1} \in F(B, A)$ such that

$$f^{-1}f = i_A \quad \text{and} \quad ff^{-1} = i_B.$$

If f^{-1} exists then it is called the **inverse** function of f .

Question: *What is the inverse of f^{-1} ?*

REMARK 39. f is invertible if and only if f^{-1} is invertible.

EXAMPLE 40. Discuss the inverse of a function defined by $f(x) = \sin x$.

PROPOSITION 41. *The inverse function is unique.*

Proof.

EXAMPLE 42. *The function $f : \mathbb{R} - \{2\} \rightarrow \mathbb{R} - \{3\}$ defined by $f(x) = \frac{3x}{x-2}$ is known to be bijective (see Example 30, Section 3.2). Determine the inverse function.*

REMARK 43. Finding the inverse of a bijective function is not always possible by algebraic manipulations. For example,

if $f(x) = e^x$ then $f^{-1}(x) = \underline{\hspace{2cm}}$

The function $f(x) = 3x^5 + 5x^3 + 2x + 2016$ is known to be bijective, but there is no way to find expression for its inverse.

THEOREM 44. *A function $f \in F(A, B)$ is invertible if and only if f is bijective.*

COROLLARY 45. *If a function $f \in F(A, B)$ is bijective, so is f^{-1} .*

Proof