

Lecture 4
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We are starting a 4 week excursion into graph theory, a subject that I am assuming no one has studied before. If you have, then you are just a bit ahead of the others and no harm has been done. Since there are good readings with all the basic definitions in them, we didn't start our book with those basic definitions. In particular, you need to have read reading number 6 before starting in on the study guide.

The big problem that is a goal of this chapter is for you to prove the 5 color theorem, namely, that any planar map can be colored with at most 5 colors so that no two countries sharing a common boundary line are the same color. Along the way to that theorem, we will learn some other major results such as there are only 5 regular platonic solids.

On exercise 3.2, you might want to use some of the information in the paragraph after it. In particular, you might need to understand about a bridge. A bridge is an edge that, if removed, breaks the graph into two pieces. To be more precise, a connected component is a piece of a graph that one can get from every vertex to another vertex along a sequence of edges, basically what I called a piece in the preceding sentence. So what does a bridge have to do with the converse of Euler's condition? If you cross a bridge, you can't get back to the other component. It's now your job to give an algorithm and show it works, namely, that you can always make the choices your algorithm dictates.

Remember to show work whenever possible on the exercises and the problems. For example, prove your answer on Exercise 3.3, Problem 1.1 part 2, and Problem 3.4 part 3.

On Problem 3.3 you will probably want to use Problem 3.2. Here's a proof of Turan's extremal theorem.

Proof: The basis step is trivial: a graph with two vertices has at most one edge. Suppose the result is known for a graph with $2k$ vertices, and consider a graph with $2(k + 1)$ vertices and no triangles. If there are no edges in this graph, we are done. Otherwise, take two vertices v_1 and v_2 that are joined by an edge. The subgraph on the remaining vertices has at most k^2 edges,

by the induction hypothesis. Because the whole graph has no triangle, the vertices v_1 and v_2 cannot both be joined by an edge to the same vertex. Hence, in addition to the k^2 edges in the subgraph, there are at most $2k$ edges from either of the distinguished vertices to the remaining vertices, and there is one edge joining v_1 to v_2 . So there are all together at most $k^2 + 2k + 1$ edges. Since $k^2 + 2k + 1 = (k + 1)^2$, the result holds for all k by induction.

The upper bound is attained for the complete bipartite graph $K_{k,k}$.

As always, feel free to email me with questions.