CHAPTER 16 – IDENTIFICATION NUMBERS

Consider the UPC code on a can of RO★TEL tomatoes

The scanner is not working so the clerk enters the numbers by hand as

$\underbrace{0\ 64144\ 28263}\ 2$

check digit

The computer recognizes this as an error. What happened? The check digit caught the error

The UPC codes use a ***check digit*** to minimize scanning errors. A check digit is a digit included in a code to help detect errors.

For the UPC code $a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}$, the check digit, a_{12}, is chosen so that S a multiple of 10 where

$$S = 3a_1 + a_2 + 3a_3 + a_4 + 3a_5 + a_6 + 3a_7 + a_8 + 3a_9 + a_{10} + 3a_{11} + a_{12}.$$

check digit

What is the check digit for a can of Ranch Style Beans if the first eleven digits are 0 46900 00108?

$$S = 3(0) + (4) + 3(6) + (9) + 3(0) + (0) + 3(0) + (0) + 3(1) + (0) + 3(8) + x$$
$$= 0 + 4 + 18 + 9 + 0 + 0 + 0 + 0 + 3 + 0 + 24 + x$$
$$= 58 + x$$

S needs to be a multiple of 10. The next multiple of 10 after 58 is 60 so

$$\begin{aligned} 58 + x &= 60 \\ -58 \quad &\ \ -58 \\ \hline x &= 2 \end{aligned}$$

So the check digit is 2.

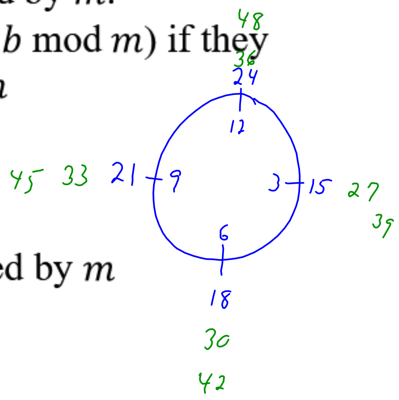
When talking about check digits, modular arithmetic will be helpful.

Definition: Congruence Modulo m

Let a, b, and m be integers with $m \geq 2$.

- Then b mod m is the remainder when b is divided by m.
- a and b are congruent modulo m (written as $a \equiv b \bmod m$) if they both have the same remainder when divided by m
  - This can also means that
    - $a - b$ is divisible by m,
    - $a - b$ is a multiple of m, and
    - $a - b$ has a remainder of 0 when divided by m

Clock is mod 12



Find the following values:

(a) 37 mod 5 = 2

$5\overline{)37}$ = 7, -35, 2

(b) 73 mod 11 = 7

$11\overline{)73}$ = 6, -66, 7

(c) 11 mod 15 = 11

$15\overline{)11}$ = 0, -0, 11

(d) 18 mod 2 = 0

$2\overline{)18}$ = 9, -18, 0

Determine if the congruences below are true or false:

$28 \equiv 3 \bmod 5$

$28-3=25$

True

$5\overline{)25}$ = 5, -25, 0 ← remainder is 0

$98 \equiv 56 \bmod 9$

$98-56=42$

False

$9\overline{)42}$ = 4, -36, 6 ← Remainder Not zero

$32 \equiv 3 \bmod 6$

$32-3=29$

False

$6\overline{)29}$ = 4, -24, 5 ← Not zero

$46 \equiv 22 \bmod 12$

$46-22=24$

True

$12\overline{)24}$ = 2, -24, 0 ← remainder is 0

Notice:

$12\overline{)46}$ = 3, -36, 10

$12\overline{)22}$ = 1, -12, 10

$10 \longleftrightarrow 10$ same #

Some types of errors when dealing with identification numbers are

- Replacing one digit with a different digit (single digit error)
- Transposing two adjacent digits (adjacent transposition error)
- Transposing two digits that are separated by another digit (jump transposition error)

Assume that the correct code was 5678 and provide an example of these errors:

Single digit error: Ex. 7678 5478 5618 5670

Adjacent Transposition Error: Ex: 6578 5768 5687

Jump Transposition Error: Ex: 7658 5876

Note that some of the digits in the UPC code are multiplied by 3. Those digits had a ***weight*** of 3. Other codes use different weights.

A code $a_1a_2a_3a_4a_5$ uses the last digit as a check digit. The check digit is found using the formula

$$a_5 = (a_1 + 7a_2 + a_3 + 7a_4) \bmod 8$$

(a) What is the check digit for the code 5384?

$a_5 = [(5)+7(3)+(8)+7(4)] \bmod 8$

$= [5 + 21 + 8 + 28] \bmod 8$

$= 62 \bmod 8 = 6$

$8\overline{)62}$ 7

-56

6

(b) Find the value of the missing digit x in the code 428x3

a_5

$3 = [(4)+7(2)+(8)+7(x)] \bmod 8$

$3 = [4 + 14 + 8 + 7x] \bmod 8$

$3 = [26+7x] \bmod 8$

when $x =$	0	1	2	3	4	5	6	7	8	9
then $26+7x =$	26	33	40	47	54	61	68	75	82	89
Sum mod 8 =	2	1	0	7	6	5	4	3	2	1

check digit

$8\overline{)26}$ 3

-24

2

$8\overline{)33}$ 4

-32

1

our check digit is unique, so

$x = 7$

Check digits are designed to catch errors, but unfortunately, in some cases, not all errors will be caught. So how do we determine which errors will ***NOT*** be caught.

KEY IDEA: For an error *not* to be caught, the correct number and the number with an error must produce the *same* check digit.

1. Write a general form of the correct number. Label the correct number with $a_1a_2a_3$ ... where each a_i represents a single digit. (Note: A digit is some whole number 0, 1,…,9.)
2. Write a general form of the incorrect number.
    a. For transposition errors, we will rearrange the a_i according to the type of transposition.
    b. For single digit errors, write the number using e_i to represent the error for the digit that should have been a_i.
3. Find the check digits for the correct number and incorrect number mod m.
4. The error will NOT be caught if the check digits are the same mod m. Use the fact that if $x \equiv y \bmod m$, then $x - y$ is a multiple of m.
5. Determine the multiples that are integers (whole numbers) between 1 and 9.
6. The error will not be caught when $|x - y|$ takes on these values.
7. List the pairs of digits where this occurs.

Let's continue with the same code we were using above to see which errors will not be caught.

Reminder: A code $a_1a_2a_3a_4a_5$ uses the last digit as a check digit. The check digit is found using the formula

$$a_5 = (a_1 + 7a_2 + a_3 + 7a_4) \bmod 8$$

(c) Will this code find an error if a single digit is entered incorrectly?

Let's look at an error in the first digit, a_1.

Correct Code: $a_1a_2a_3a_4a_5$

Incorrect Code: $e_1a_2a_3a_4a_5$

So the check digit for the correct code is:

$(a_1 + 7a_2 + a_3 + 7a_4) \text{ mod } 8$

and the check digit for the incorrect code is:

$(e_1 + 7a_2 + a_3 + 7a_4) \text{ mod } 8$

The error will NOT be caught if the check digits are the same which means

$(a_1 + 7a_2 + a_3 + 7a_4) - (e_1 + 7a_2 + a_3 + 7a_4)$ is a multiple of 8.

$a_1 + \cancel{7a_2} + \cancel{a_3} + \cancel{7a_4} - e_1 - \cancel{7a_2} - \cancel{a_3} - \cancel{7a_4}$ is a multiple of 8

$a_1 - e_1$ is a multiple of 8

This simplifies to

$a_1 - e_1$ is a multiple of 8.

Digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, so the difference between digits is always an integer and digits can never be separated by more than 9. If the difference between a_1 and e_1 is 0, we didn't really make a mistake, so we want to find the integer multiples of 8 between 1 and 9.

Mul of 8 are $\{0, 8, 16, 24 ...\}$

What are the integer multiples of 8 between 1 and 9? 8

So if $|a_1 - e_1| = 8$, the error will not be caught.

When does this occur? What pair(s) of digits are separated by 8 units?

0 and 8 1 and 9

The same logic also applies to the third digit.

Therefore, we need to check the even-numbered positions.

Let's look at an error in the second digit, a_2.

Correct Code: $a_1a_2a_3a_4a_5$

Incorrect Code: $a_1e_2a_3a_4a_5$

So the check digit for the correct code is:

$(a_1 + 7a_2 + a_3 + 7a_4) \bmod 8$

and the check digit for the incorrect code is:

$(a_1 + 7e_2 + a_3 + 7a_4) \bmod 8$

The error will NOT be caught if the check digits are the same which means

$(a_1 + 7a_2 + a_3 + 7a_4) - (a_1 + 7e_2 + a_3 + 7a_4)$ is a multiple of 8.

$\cancel{a_1} + 7a_2 + \cancel{a_3} + \cancel{7a_4} - \cancel{a_1} - 7e_2 - \cancel{a_3} - \cancel{7a_4}$ is a multiple of 8

$7a_2 - 7e_2$ is a multiple of 8

This simplifies to

$7a_2 - 7e_2$ is a multiple of 8,

which means $7(a_2 - e_2)$ is a multiple of 8, $\frac{56}{7} = 8$

which means, $a_2 - e_2$ is a multiple of $8/7$. bigger than 10

Multiples of $\frac{8}{7}$ are $\pm\left\{0, \frac{8}{7}, \frac{16}{7}, \frac{24}{7}, \frac{32}{7}, \frac{40}{7}, \frac{48}{7}, \frac{56}{7}, \frac{64}{7}, \frac{72}{7}, \frac{80}{7}, \dots\right\}$ so

what are the integer multiples of $\frac{8}{7}$ between 1 and 9? 8

So if $|a_2 - e_2| =$ 8, the error will not be caught.

For what pair(s) of digits does this occur?

0 and 8 1 and 9

The same logic also applies to the fourth digit.

We have now checked all four digits for single-digit errors and have determined that the following single-digit errors will not be caught.

when 0 and 8 are interchanged [0 entered for 8 or 8 entered for 0]

and when 1 and 9 are interchanged

(d) Will this code find all adjacent transposition errors?

Let's look at an adjacent transposition of the first two digits, a_1 and a_2.

Correct Code: $a_1a_2a_3a_4a_5$

Incorrect Code: $a_2a_1a_3a_4a_5$

So the check digit for the correct code is:

$(a_1 + 7a_2 + a_3 + 7a_4) \bmod 8$

and the check digit for the incorrect code is:

$(a_2 + 7a_1 + a_3 + 7a_4) \bmod 8$

The error will NOT be caught if the check digits are the same which means

$(a_1 + 7a_2 + a_3 + 7a_4) - (a_2 + 7a_1 + a_3 + 7a_4)$ is a multiple of 8.

$\underline{a_1} + \underline{\underline{7a_2}} + \cancel{a_3} + \cancel{7a_4} \underline{\underline{- a_2}} \underline{- 7a_1} - \cancel{a_3} - \cancel{7a_4}$ is a multiple of 8

$-6a_1 + 6a_2$ is a multiple of 8

This simplifies to

$6a_2 - 6a_1$ is a multiple of 8,

which means $6(a_2 - a_1)$ is a multiple of 8,

which means, $a_2 - a_1$ is a multiple of $\frac{8}{6} = \frac{4}{3}$.

Multiples of $\frac{4}{3}$ are $\pm\left\{0, \frac{4}{3}, \frac{8}{3}, \underset{=4}{\boxed{\frac{12}{3}}}, \frac{16}{3}, \frac{20}{3}, \underset{=8}{\boxed{\frac{24}{3}}}, \frac{28}{3}, \underset{\uparrow > 10}{\frac{32}{3}}, \ldots\right\}$ so

what are the integer multiples of $\frac{4}{3}$ between 1 and 9? 4 and 8

So if $|a_2 - a_1| = 4$ or 8, the error will not be caught.

For what pair(s) of digits does this occur?

and

0 & 4 0 & 8 and

1 & 5 1 & 9

2 & 6

3 & 7

4 & 8

5 & 9

The same logic also applies to the remaining adjacent digits.

(e) Will this code find all jump transposition errors?

Let's check a_1 and a_3 transposed

Correct code: $a_1 a_2 a_3 a_4 a_5$ Incorrect code: $a_3 a_2 a_1 a_4 a_5$

Check digit for correct code is $(a_1 + 7a_2 + a_3 + 7a_4) \bmod 8$

Check digit for incorrect code is $(a_3 + 7a_2 + a_1 + 7a_4) \bmod 8$

The error will not be caught if the check digits are the same which means

$(a_1 + 7a_2 + a_3 + 7a_4) - (a_3 + 7a_2 + a_1 + 7a_4)$ is a multiple of 8

$\cancel{a_1} + \cancel{7a_2} + \cancel{a_3} + \cancel{7a_4} - \cancel{a_3} - \cancel{7a_2} - \cancel{a_1} - \cancel{7a_4}$ is a multiple of 8

0 is a multiple of 8

0 is a multiple of 8 is always true so jump transpositions of the first and 3rd digits will always NOT be caught.

So NO jump transpositions between the 1st and 3rd digits will be caught.

why is this??? Because the numbers switched had the same weight, the sum for the correct check digit and the incorrect check digit will be the same, so the check digits will be the same.

The same reasoning will apply to transposing the 2nd and 4th digits, So this scheme does not catch ANY jump transpositions

Data can be encoded in identification numbers.

An Illinois driver's license contains a portion (Y-YDDD) that represents the last two digits of the year of birth of the driver and codes the birthday according to the following formulas where m represents birth month and d represents the birth date.

$$\text{Male} = 31(m-1)+d$$
$$\text{Female} = 31(m-1)+d+600$$

(a) What would the Y-YDDD digits of an of Illinois driver's license number look like for a man born on April 23, 1972?

Y-Y = 7-2

Male $= 31(m-1)+d$ where $m=4$ $d=23$

$= 31(4-1)+23$

$= 31(3)+23$

$= 93+23 = 116$

So Y-YDDD is 7-2116

(b) What do you know about a person whose Y-YDDD digits are 0-1866?

Female because DDD > 600

Y-Y = 0-1 so born in 2001.

$866 = 31(m-1)+d+600$

$-600 \qquad -600$

$266 = 31(m-1)+d$

$8 \leftarrow$ 8 full months have passed, so we are in month 9

$31\overline{)266}$

-248

$18 \leftarrow$ days into that month

So She was born Sept. 18, 2001

(c) What do you know about a person whose Y-YDDD digits are 8-1199?

Male because DDD < 600

Y-Y = 8-1 so born in 1981

$199 = 31(m-1)+d$

$6 \leftarrow$ 6 full months have passed, so we are in month 7

$31\overline{)199}$

-186

$13 \leftarrow$ days into that month

He was born July 13, 1981

A car rental company numbers its reservations with the ID number followed by a check digit where the check digit is the ID number mod 7. Is the reservation number 7645882 a valid car rental reservation number?

check digit

$$\begin{array}{r} 109226 \\ 7\overline{)764588} \\ -764582 \\ \hline 6 \end{array}$$

6 ← 6 is the check digit for ID 764588 which does not match

so this is NOT a valid car rental reservation number.

Credit cards have 15-digit numbers with a check digit in position 16.

Let D = the sum of the digits in odd-numbered positions

Let E = the sum of the digits in even-numbered positions

(not including the check digit)

Let T = the number of digits in odd-numbered positions that are larger than 4.

Let $C = 2D + E + T + a_{16}$

The check digit, a_{16}, is chosen so C is a multiple of 10.

The number 5213 7512 3412 3456 was listed on a credit card advertisement. Is the check digit correct?

$D = 5+1+7+1+3+1+3+5 = 26$

$E = 2+3+5+2+4+2+4 = 22$

$T = 3$

$C = 2D + E + T + a_{16}$

$= 2(26) + (22) + 3 + 6$

$= 83$

C should be a multiple of 10. It is not, so this is not a valid credit card number

If we wanted to find the check digit for 5213 7512 3412 345x, then

$C = 2D + E + T + x$

$= 2(26) + (22) + 3 + x$

$= 77 + x$

C should be a multiple of 10. The next multiple of 10 is 80, so

$$\begin{array}{rcl} 80 &=& 77 + x \\ -77 & & -77 \\ \hline 3 &=& x \end{array}$$

SAMPLE EXAM QUESTIONS FROM CHAPTER 16

So check digit is not part of

1. Determine the check digit that should be appended to the identification number 572638, if the check digit is the number needed to bring the total of all the digits to a multiple of 10.

(A) The code is invalid [✓ (B) 9] (C) 1 (D) 2

(E) None of these

5+7+2+6+3+8 = 31

The next multiple of 10 is 40, so

check digit

$$\begin{array}{rcl} 40 &=& 31 + x \\ -31 && -31 \\ \hline 9 &=& x \end{array}$$

2. Which, if any, of the statements below are true? Mark all correct answers.

(A) 103 ≡ 1 mod 4

103 - 1 = 102 False

$$\begin{array}{r} 25 \\ 4\overline{)102} \\ -100 \\ \hline 2 \end{array}$$

[✓ (B)] 79 ≡ 2 mod 11

79 - 2 = 77 True

$$\begin{array}{r} 7 \\ 11\overline{)77} \\ -77 \\ \hline 0 \end{array}$$

[✓ (C)] 49 ≡ 13 mod 12

49 - 13 = 36 True

$$\begin{array}{r} 3 \\ 12\overline{)36} \\ -36 \\ \hline 0 \end{array}$$

(D) 38 ≡ 4 mod 7

38 - 4 = 34 False

$$\begin{array}{r} 4 \\ 7\overline{)34} \\ -28 \\ \hline 6 \end{array}$$

remainder not zero

remainder of zero

(E) None of these are true.

3. The number 4320 is accidentally entered as 2340. What type of error is this?

(A) A transposition error

[✓ (B)] A jump transposition error

(C) A single digit error

(D) A baseball error

(E) None of these

4. The last three digits of a person's ID are calculated based on their birthday where *m* represents birth month and *d* represents the birth date.

$$\text{Male} = 35(m-1)+d$$

$$\text{Female} = 35(m-1)+d+500$$

(a) What are the last three digits of a man's ID number if he was born on July 4th?

Male $= 35(m-1)+d$ where $m=7$ $d=4$

$= 35(7-1) + 4$

$= 35(6) + 4$

$= 210 + 4$

$= \underline{\underline{214}}$

(b) What do you know about a person if the last three digits of the person's ID number are 585?

Female because $585 > 500$

$585 = 35(m-1)+d+500$

$-500 \qquad\qquad -500$

$85 = 35(m-1)+d$

She was born March 15

$$35\overline{)85}\ \text{quotient } 2$$

-70

15

2 ← 2 Full months have passed, So we are in the 3rd month

15 ← 15 days into that month

(c) What do you know about a person if the last three digits of the person's ID number is 175?

Male because $175 < 500$

$175 = 35(m-1)+d$

Not a valid ID number

$$35\overline{)175}\ \text{quotient } 5$$

-175

0

5 ← 5 Full months have passed So we are the 6th month.

0 ← 0 days into that month

June 1 would be

$35(6-1)+1$

$35(5)+1$

$175 + 1 = 176$

May 31 would be

$35(5-1)+31$

$35(4)+31$

$140 + 31 = 171$

5. A code is given by $a_1a_2a_3a_4$ where a_4 is the check digit. The check digit is $a_4 = 2a_1 + 5a_2 + 7a_3 \bmod 9$.

(a) Determine the value of x in the code $4x83$, given that the check digit is valid.

check digit a_4

4483

$$3 = [2(4) + 5(x) + 7(8)] \bmod 9$$

$$3 = [8 + 5x + 56] \bmod 9$$

$$3 = [64 + 5x] \bmod 9$$

if $x =$	0	1	2	3	4	5	6	7	8	9
$64+5x =$	64	69	74	79	84	89	94	99	104	109
check digit is sum mod 9 $=$	1	6	2	7	3	8	4	9	5	1

The only one with a check digit of 3 was $x=4$

$$\begin{array}{r} 7 \\ 9\overline{)64} \\ -63 \\ \hline 1 \end{array}$$

(b) Determine if the check digit will find all single digit errors in the second position.

Correct code: $a_1a_2a_3a_4$ Incorrect code: $a_1e_2a_3a_4$

Check digit for the correct code: $(2a_1 + 5a_2 + 7a_3) \bmod 9$

Check digit for incorrect code: $(2a_1 + 5e_2 + 7a_3) \bmod 9$

The error will not be caught if the check digits are the same which means

$(2a_1 + 5a_2 + 7a_3) - (2a_1 + 5e_2 + 7a_3)$ is a multiple of 9

$\cancel{2a_1} + 5a_2 + \cancel{7a_3} - \cancel{2a_1} - 5e_2 - \cancel{7a_3}$ is a multiple of 9

$5a_2 - 5e_2$ is a multiple of 9

$5(a_2 - e_2)$ is a multiple of 9

$a_2 - e_2$ is a multiple of $\frac{9}{5}$

Multiples of $\frac{9}{5}$ are $\pm\left\{0, \frac{9}{5}, \frac{18}{5}, \frac{27}{5}, \frac{36}{5}, \overset{9}{\boxed{\frac{45}{5}}}, \underbrace{\frac{54}{5}, \frac{63}{5}, \frac{72}{5}, \frac{81}{5}, \frac{90}{5}, \ldots}_{}\right\}$ bigger than 10

Integer multiple(s) of $\frac{9}{5}$ between 1 and 9 is/are 9

So if $|a_2 - e_2| = 9$, the error will not be caught.

This occurs when 0 and 9 are interchanged

(c) Determine if the check digit will find all transposition errors in the second and third positions.

Correct code: $a_1 a_2 a_3 a_4$ Incorrect code: $a_1 a_3 a_2 a_4$

Check digit for correct code is $(2a_1 + 5a_2 + 7a_3) \bmod 9$

Check digit for incorrect code is $(2a_1 + 5a_3 + 7a_2) \bmod 9$

The error will not be caught if the check digits are the same, which means

$(2a_1 + 5a_2 + 7a_3) - (2a_1 + 5a_3 + 7a_2)$ is a multiple of 9

$\cancel{2a_1} + 5a_2 + 7a_3 - \cancel{2a_1} - 5a_3 - 7a_2$ is a multiple of 9

$-2a_2 + 2a_3$ is a multiple of 9

$2a_3 - 2a_2$ is a multiple of 9

$2(a_3 - a_2)$ is a multiple of 9

$a_3 - a_2$ is a multiple of $\frac{9}{2}$

Multiples of $\frac{9}{2}$ are $\pm\{0, \frac{9}{2}, \underset{=9}{\boxed{\frac{18}{2}}}, \frac{27}{2}, \dots\}$

↖ bigger than 10

So the integer multiple(s) of $\frac{9}{2}$ between 1 and 9 is/are 9

So if $|a_3 - a_2| = 9$, the error will not be caught.

This occurs when 0 and 9 are interchanged.