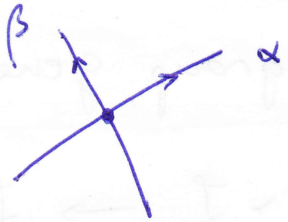
Goldman bracket — Lecture 1

Σ smooth, oriented surface (w/ or w/out ∂).

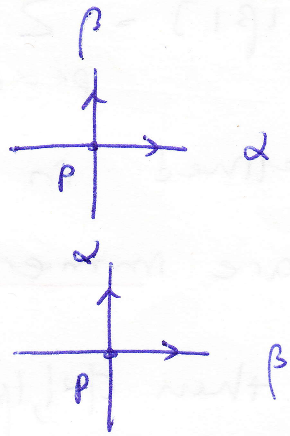
α, β immersed loops in Σ s.t. $\alpha \cap \beta$ are at worst <u>double points</u>.

$\alpha, \beta : S^1 \to \Sigma$

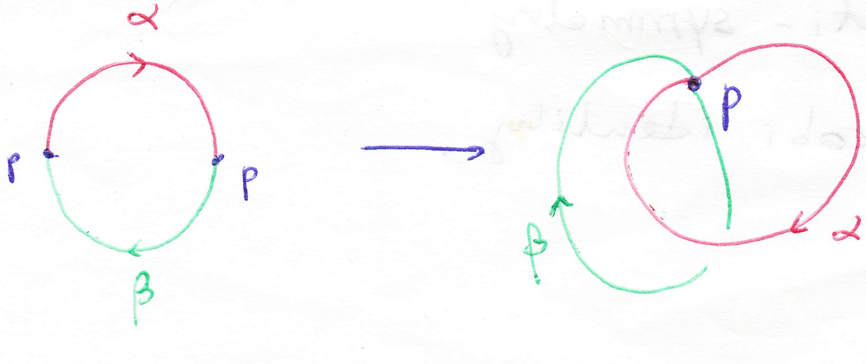


<u>Def</u>: $p \in \alpha \cap \beta$

① $\varepsilon(p; \alpha, \beta) = \begin{cases} 1, \\ -1, \end{cases}$



② $\alpha, \beta : S^1 \to \Sigma$



3. make sense of

$$[\alpha, \beta] = \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta)\, \alpha_p \beta$$

$\hat{\pi}$ = { free homotopy class $|\alpha|$ of $\alpha : S^1 \to \Sigma$ }

$\mathcal{L}$ = free abelian group generated by $\hat{\pi}$.

~~Thm 1~~ Def: $[\ ,\ ] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ by

$$[|\alpha|, |\beta|] = \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta)\, |\alpha_p \beta|$$

Thm: $(\mathcal{L}, [\ ,\ ])$ is well defined in the sense that if α, α', β and β' are immersed loops s.t. $\alpha \sim \alpha'$ and $\beta \sim \beta'$, then $[|\alpha|, |\beta|] = [|\alpha'|, |\beta'|]$.

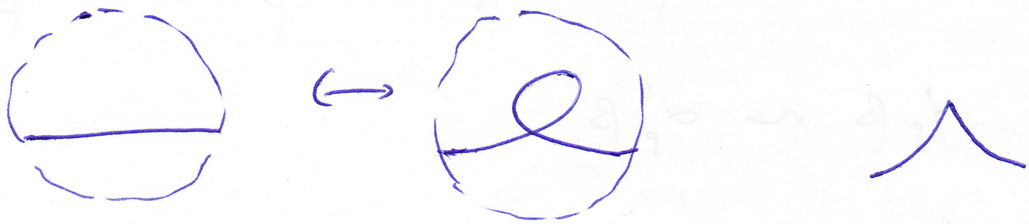
$\sim$: freely homotopic to

Thm 2. $(\mathcal{L}, [\ ,\ ])$ is a Lie algebra, i.e.,

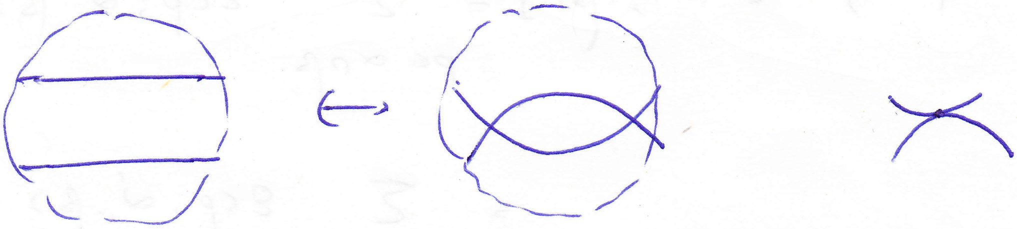
1. anti-symmetry
2. Jacobi identity.

Lemma: If α and β are immersed loops s.t. $\alpha \sim \beta$, then α and β are related by isotopis and the composition of the following three moves.

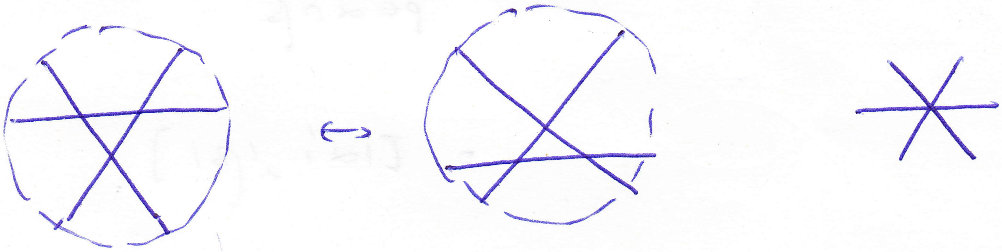
(m1)



(m2)



(m3)

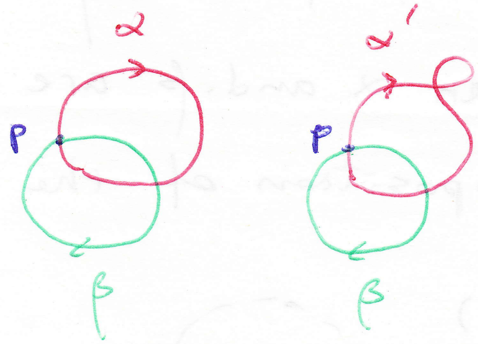


Pf of thm 1:

It suffices to check that $[|\alpha|, |\beta|] = [|\alpha'|, |\beta|]$ if α and α' are related by m1, m2 ~~and~~ or m3.

For (m1) we have

$$\alpha \cap \beta = \alpha' \cap \beta$$



and $\forall\ p \in \alpha \cap \beta = \alpha' \cap \beta$,

$$\alpha_p \beta \sim \alpha'_p \beta .$$

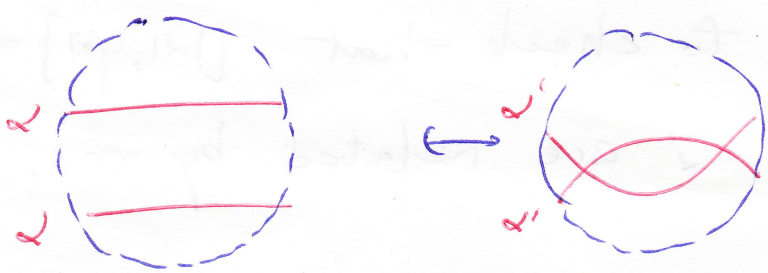
Therefore, $$[|\alpha|, |\beta|] = \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha . \beta) |\alpha_p \beta|$$

$$= \sum_{p \in \alpha' \cap \beta} \varepsilon(p; \alpha' . \beta) |\alpha'_p \beta|$$

$$= [|\alpha'|, |\beta|] .$$

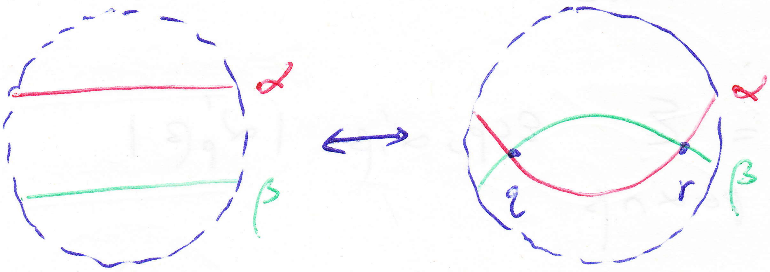
For (m2), there are two cases:

Case 1:



Similar to m1.

Case 2:



claim: ① $\varepsilon(q;\alpha',\beta) = -\varepsilon(r;\alpha',\beta)$

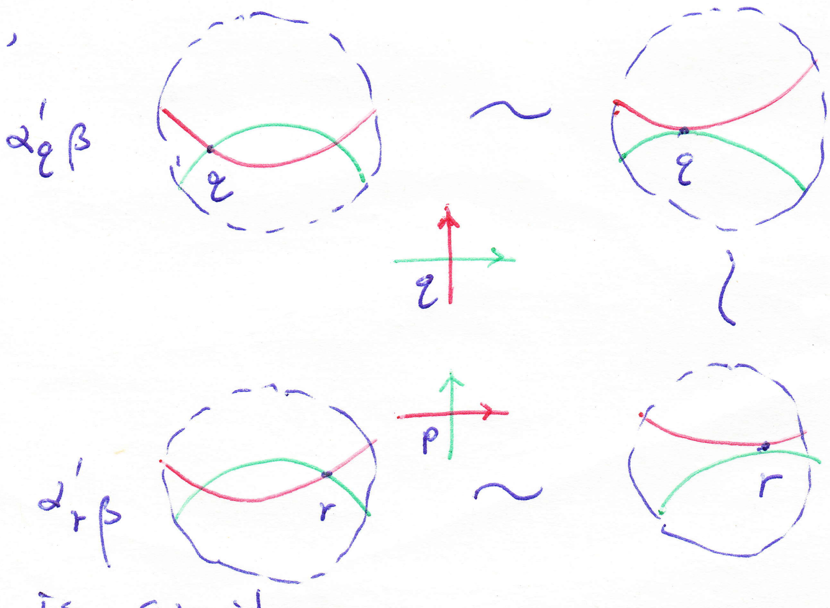
and $\alpha'_q\beta \sim \alpha'_r\beta$.

② $\forall p \in (\alpha' \cap \beta \setminus \{q,r\})$,

$\varepsilon(p;\alpha',\beta) = \varepsilon(p;\alpha,\beta)$ and

$\alpha'_p\beta \sim \alpha_p\beta$.

For ①,



② is similar.

Therefore, we have

$$[|\alpha'|, |\beta|] = \sum_{p \in \alpha' \cap \beta} \varepsilon(p; \alpha', \beta) \, |\alpha'_p \beta|$$

$$= \sum_{p \in (\alpha' \cap \beta \setminus \{q, r\})} \varepsilon(p; \alpha', \beta) \, |\alpha'_p \beta|$$

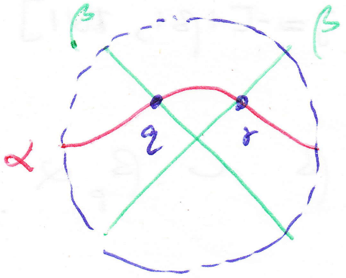
$$+ \varepsilon(q; \alpha', \beta) |\alpha'_q \beta| + \varepsilon(r; \alpha', \beta) |\alpha'_r \beta|$$

$$\overset{①②}{=} \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta) \, |\alpha_p \beta|$$

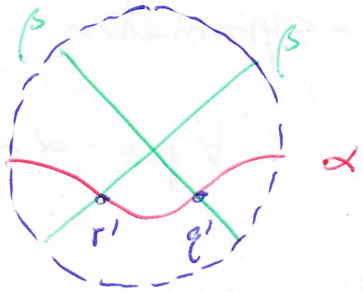
$$= [|\alpha|, |\beta|].$$

For (m3), there are two cases.

Case 1



$\longleftrightarrow$



claim: ① If $p \in \{q, r\}$, then

$\varepsilon(p'; \alpha', \beta) = \varepsilon(p; \alpha, \beta)$ and

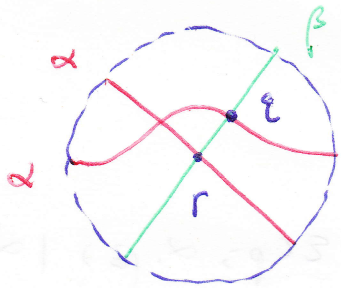
$\alpha'_{p'}\beta \sim \alpha_p \beta$.

② $\forall p \in \alpha \cap \beta \setminus \{q, r\}$,

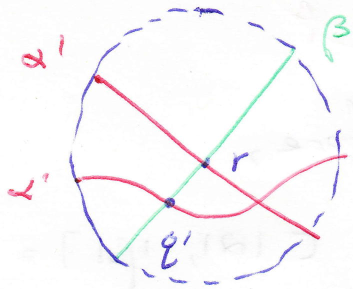
$\varepsilon(p; \alpha', \beta) = \varepsilon(p; \alpha, \beta)$ and

$\alpha'_p \beta \sim \alpha_p \beta$.

Case 2



$\longleftrightarrow$



claim: ① $\varepsilon(q'; \alpha', \beta) = \varepsilon(q; \alpha, \beta)$ and

$\alpha'_{q'} \beta \sim \alpha_q \beta$.

② $\forall p \in \alpha \cap \beta \setminus \{q\}$ (including r),

$\varepsilon(p; \alpha', \beta) = \varepsilon(p; \alpha, \beta)$ and

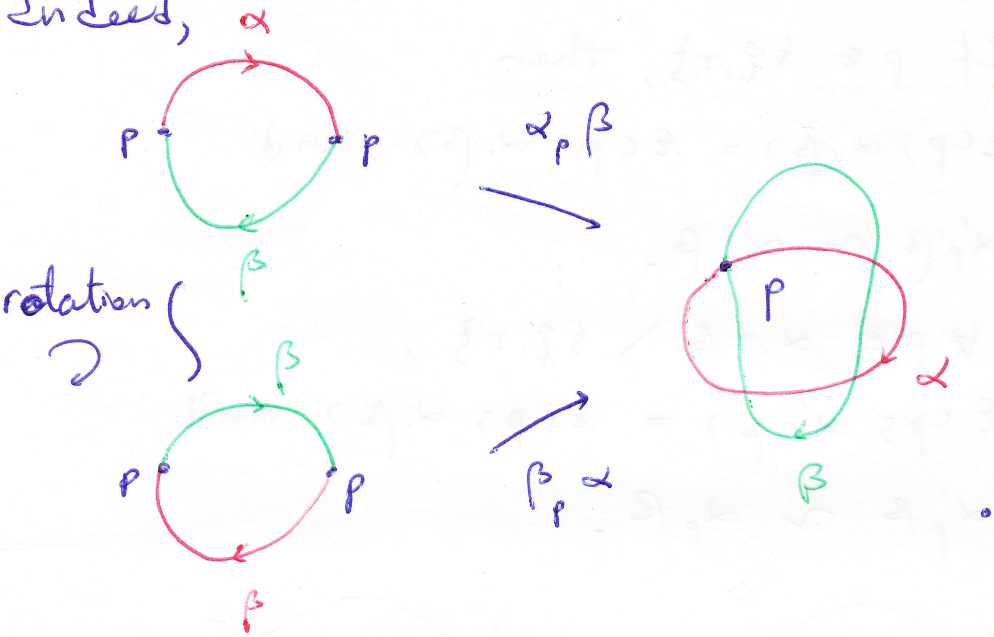
$\alpha'_p \beta \sim \alpha_p \beta$.

Pf of thm 2 :

① Anti-symmetry $[|\alpha|, |\beta|] = -[|\beta|, |\alpha|]$.

claim: $\forall p \in \alpha \cap \beta$, $\alpha_p \beta \sim \beta_p \alpha$.

Indeed,



Therefore,

$$[|\alpha|, |\beta|] = \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta)\, |\alpha_p \beta|$$

$$= \sum_{p \in \beta \cap \alpha} -\varepsilon(p; \beta, \alpha)\, |\beta_p \alpha|$$

$$= -[|\beta|, |\alpha|].$$

Jacobi identity:

$$[[|\alpha|,|\beta|],|\gamma|]+[[|\beta|,|\gamma|],|\alpha|]+[[|\gamma|,|\alpha|],|\beta|]=0.$$

By definition,

$$[[|\alpha|,|\beta|],|\gamma|] = [\sum_{p\in\alpha\cap\beta} \varepsilon(p;\alpha,\beta)\,|\alpha_p\beta|,|\gamma|]$$

$$= \sum_{q\in\alpha_p\beta\cap\gamma}\ \sum_{p\in\alpha\cap\beta} \varepsilon(q;\alpha_p\beta,\gamma)\,\varepsilon(p;\alpha,\beta)\,(\alpha_p\beta)_q\gamma$$

$\underline{\underline{\alpha_p\beta\cap\gamma=(\alpha\cap\gamma)\cup(\beta\cap\gamma)}}$

$$\sum_{q\in\alpha\cap\gamma}\ \sum_{p\in\alpha\cap\beta} \varepsilon(q;\alpha,\gamma)\,\varepsilon(p;\alpha,\beta)\,\underline{(\alpha_p\beta)_q\gamma}$$

$$+\sum_{r\in\beta\cap\gamma}\ \sum_{p\in\alpha\cap\beta} \varepsilon(r;\beta,\gamma)\,\varepsilon(p;\alpha,\beta)\,\underline{(\alpha_p\beta)_r\gamma}$$

Similarly,

$$[[|\beta|,|\gamma|],|\alpha|] = \sum_{p\in\beta\cap\alpha}\ \sum_{r\in\beta\cap\gamma} \varepsilon(p;\beta,\alpha)\,\varepsilon(r;\beta,\gamma)\,\underline{(\beta_r\gamma)_p\alpha}$$

$$+\sum_{q\in\gamma\cap\alpha}\ \sum_{r\in\beta\cap\gamma} \varepsilon(q;\gamma,\alpha)\,\varepsilon(r;\beta,\gamma)\,\underline{(\beta_r\gamma)_q\alpha}.$$

$$[[|\gamma|,|\alpha|],|\beta|] = \sum_{r\in\gamma\cap\beta}\ \sum_{q\in\gamma\cap\alpha} \varepsilon(r;\gamma,\beta)\,\varepsilon(q;\gamma,\alpha)\,\underline{(\gamma_q\alpha)_r\beta}$$

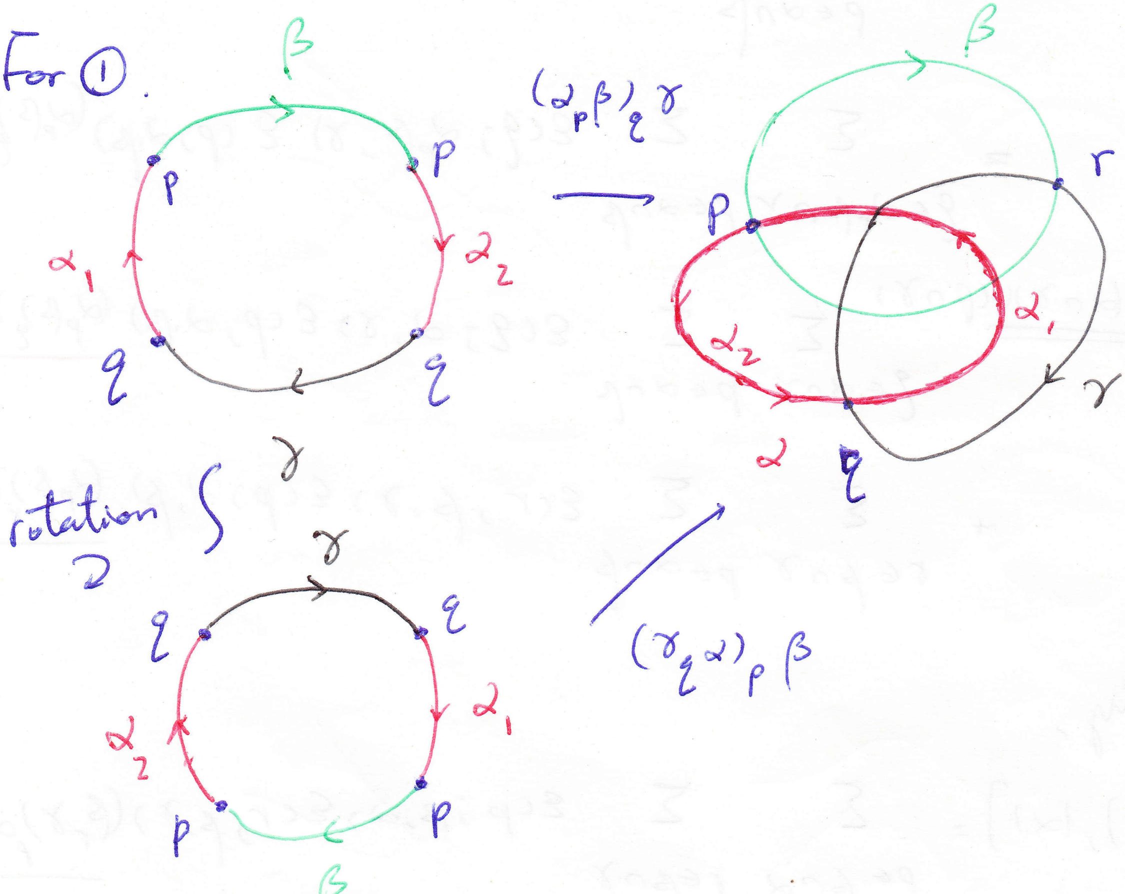
$$+\sum_{p\in\alpha\cap\beta}\ \sum_{q\in\gamma\cap\alpha} \varepsilon(p;\alpha,\beta)\,\varepsilon(q;\gamma,\alpha)\,\underline{(\gamma_q\alpha)_p\beta}.$$

claim: ① $(\alpha_p \beta)_q \gamma \sim (\gamma_q \alpha)_p \beta$,

② $(\alpha_p \beta)_r \gamma \sim (\beta_r \gamma)_p \alpha$,

③ $(\beta_r \gamma)_q \alpha \sim (\gamma_q \alpha)_r \beta$.

For ①.



② and ③ are similar.

Therefore, all the ~~th~~ terms cancel out, and the Jacobi identity holds.

Unoriented loops

~~Let $i: S^1 \to S^1 \subset \mathbb{C}$, and $\alpha^{-1} = \alpha \circ i : S^1 \to \Sigma$~~
~~$z \mapsto z^{-1}$~~

<u>Prop</u>: The map $\iota : \mathcal{L} \to \mathcal{L}$ defined by

$$|\alpha| \longmapsto |\alpha^{-1}|$$

is a Lie algebra homomorphism.

<u>Pf</u>:
$$[\iota|\alpha|, \iota|\beta|] = \sum_{p \in \alpha^{-1} \cap \beta^{-1}} \varepsilon(p; \alpha^{-1}, \beta^{-1}) \, |\alpha^{-1}{}_p \beta^{-1}|$$

$$\underset{\alpha_p\beta \sim \beta_p\alpha}{=} \sum_{p \in \alpha^{-1} \cap \beta^{-1}} \varepsilon(p; \alpha^{-1}, \beta^{-1}) \, |\beta^{-1}{}_p \alpha^{-1}|$$

$$= \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta) \, |(\alpha_p \beta)^{-1}|$$

$$= \iota [|\alpha|, |\beta|].$$

Let $\bar{\alpha} = |\alpha| + \iota|\alpha|$, then the submodule $\bar{\mathcal{L}}$ generated by $\{ \bar{\alpha} \mid |\alpha| \in \mathcal{L} \}$ is fixed by ι, hence is a Lie algebra.

- $\bar{\mathcal{L}}$ is the Lie algebra generated by the free homotopy classes of unoriented loops.

$$[\bar{\alpha}, \bar{\beta}] = [\,|\alpha| + \iota|\alpha|,\ |\beta| + \iota|\beta|\,]$$

$$= [\,|\alpha|, |\beta|\,] + [\,\iota|\alpha|, |\beta|\,] + [\,|\alpha|, \iota|\beta|\,] + [\,\iota|\alpha|, \iota|\beta|\,]$$

$$= \overline{[\,|\alpha|, |\beta|\,]} + \overline{[\,|\alpha|, |\beta^{-1}|\,]}$$

$$= \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha.\beta)\ \overline{\alpha_p \beta}$$

$$+ \sum_{p \in \alpha \cap \beta^{-1}} \varepsilon(p; \alpha, \beta^{-1})\ \overline{\alpha_p \beta^{-1}}$$

$$= \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha.\beta)\left(\overline{\alpha_p \beta} - \overline{\alpha_p \beta^{-1}}\right)$$

Thm 3: The map $[\,,\,] : \bar{\mathcal{L}} \times \bar{\mathcal{L}} \to \bar{\mathcal{L}}$ defined by

$$\boxed{[\bar{\alpha}, \bar{\beta}] = \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha.\beta)\left(\overline{\alpha_p \beta} - \overline{\alpha_p \beta^{-1}}\right)}$$

is a well-defined Lie bracket.