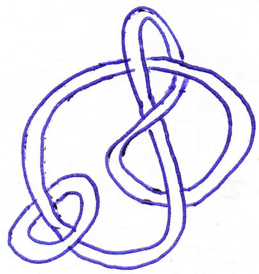
Lecture 3 ①

- M smooth, oriented 3-mfd (w/ or w/out ∂).
- A framed link in M is an embedding

$$\alpha : \coprod S^1 \times (-\varepsilon, \varepsilon) \longrightarrow M.$$



- $\mathcal{L}_M$ = Isotopy classes of framed links in M.

$\mathbb{C}\mathcal{L}_M$ = $\mathbb{C}$-vector space generated by $\mathcal{L}_M$.

$$\underset{\psi}{a} = \sum_{i=1}^{n} c_i \alpha_i, \quad c_i \in \mathbb{C}, \quad \alpha_i \in \mathcal{L}_M.$$

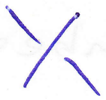
Kauffman bracket skein module (Przytycki, Turaev).

$$S_q(M) = \mathbb{C}\mathcal{L}_M[q] \,/\, ①\ ②$$

$$\underset{\psi}{a} = \sum_{i=-m}^{n} a_i q^i, \quad a_i \in \mathbb{C}\mathcal{L}_M$$

$\mathbb{C}\mathcal{L}_M[q]$ = ring of Laurent polynomials in q w/ coefficients in $\mathbb{C}\mathcal{L}_M$.

① Kauffman bracket skein relation :

 $= q$)($+ q^{-1}$ 

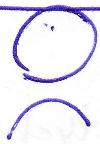
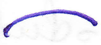
↑
vertical framing

② Framing relation :

 $= -q^{2} - q^{-2}$

-  and  differ by a twist

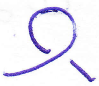
 $= -q^{3}$ 

$\overset{①}{=} q$  $+ q^{-1}$ 

$\overset{②}{=} \left(q(-q^{2} - q^{-2}) + q^{-1}\right)$ 

$= -q^{3}$ , Similarly  $= -q^{-3}$  Ex.

$q = -1$,  $=$  ~~Invariant under m~~

$q = 1$.  $= -$  ⟷ spin structure

$M = S^3$, $S_q(S^3) \multimap$ Jones polynomial

$\multimap$ Khovanov homology.

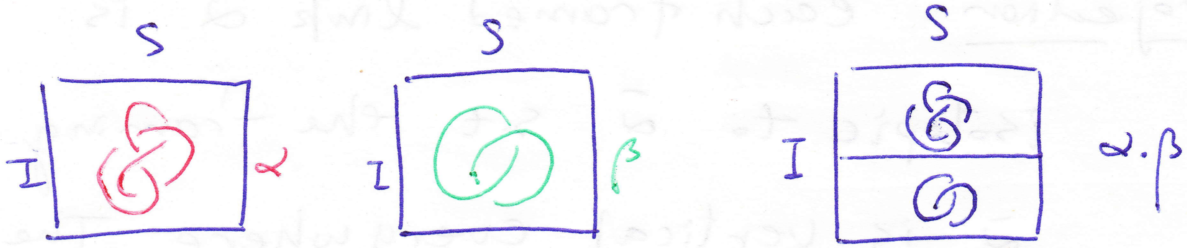
$M = \Sigma \times I$, Σ smooth, oriented closed surface.

Notation: $S_q(\Sigma) = S_q(\Sigma \times I)$.

link diagram

There is a product $\cdot$ on $S_q(\Sigma)$: $\forall\, \alpha, \beta \in \mathcal{L}_m$,

$\alpha \cdot \beta =$ "stacking α above β along I".

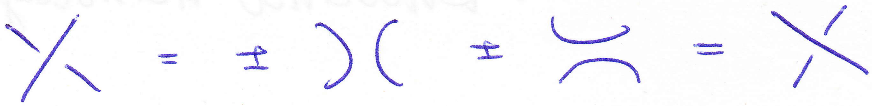


"$\cdot$" linearly extends to $\mathbb{C}\mathcal{L}_m$, then to $\mathbb{C}\mathcal{L}_m[q]$.

Thm1 (Przytycki). $(S_q(\Sigma), \cdot)$ is a well defined associative $\mathbb{C}[q]$-algebra.

pf: Let $\mathcal{I}$ be the submodule of $\mathbb{C}\mathcal{L}_m[q]$ generated by ① ②. It is easy to see that $\mathcal{I} \cdot \mathbb{C}\mathcal{L}_m[q] \subset \mathcal{I}$.

$q = \pm 1$, "$\circ$" is commutative.



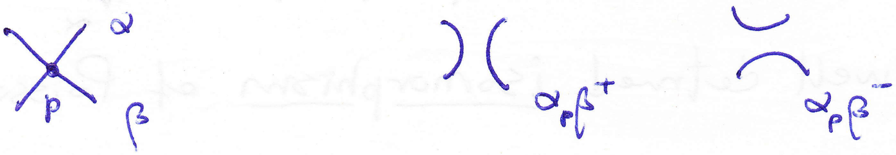
$S_{\pm 1}(\Sigma)$ would be considered as generated by the projections of framed links on Σ, that is, the regular isotopy classes of immersed loops on Σ.

projection: each framed link α is isotopic to $\bar{\alpha}$ s.t the framing of $\bar{\alpha}$ is vertical everywhere. The projection of α is $P_1 \circ \bar{\alpha}$, where $P_1 : \Sigma \times I \to \Sigma$.

regular isotopy: two immersed loops α and β are regularly isotopic to each other if they are related by isotopy and the compositions of (m2) and (m3).

There is a $\{,\}: S_{\pm 1}(\Sigma) \times S_{\pm 1}(\Sigma) \to S_{\pm 1}(\Sigma)$ by

$$\{\alpha, \beta\} = \frac{1}{2} \sum_{p \in \alpha \cap \beta} (\alpha_p \beta^+ - \alpha_p \beta^-)$$



Thm 2 (Bullock - Frohman - Kania-B)

$(S_{\pm 1}(\Sigma), \cdot, \{,\})$ is a well defined Poisson algebra.

Pf: By deformation quantization (Thm 4)

<u>Relationship w/ character variety</u>

Recall: $\pi = \pi_1(\Sigma)$, $G = SL(2, \mathbb{C})$.

$$\chi(\Sigma) = \mathrm{Hom}(\pi, G)/G$$

$\mathcal{R}(\Sigma)$ = ring of regular functions on $\chi(\Sigma)$

= $\mathrm{span}_{\mathbb{C}} \{ tr_\alpha \mid \alpha$ conjugacy classes in $\pi \}$

$$\{,\} : \mathcal{R}(\Sigma) \times \mathcal{R}(\Sigma) \longrightarrow \mathcal{R}(\Sigma)$$

$$\{tr_\alpha, tr_\beta\} = \frac{1}{2} \sum_{p \in \alpha \cap \beta} \varepsilon_p(\alpha, \beta) \left(tr_{\alpha_p \beta^-} - tr_{\alpha_p \beta^+} \right)$$

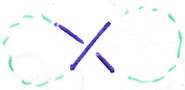
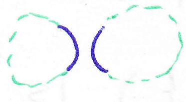
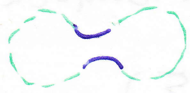
Thm 3 (Bullock, Przytycky-Sikora, Charles-Marché)

The map

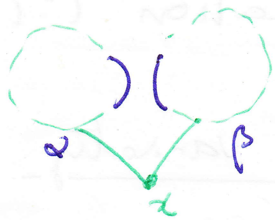
$$\Phi : S_{-1}(\Sigma) \longrightarrow \mathcal{R}(\Sigma)$$
$$\alpha \longmapsto -tr_\alpha$$

is a well defined <u>isomorphism</u> of Poisson algebras.

Pf. <u>Well definition</u>: it's to verify

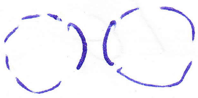
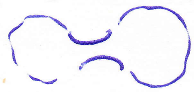
① $\Phi($  $) = \Phi(-$  $-$  $)$

② $\Phi($  $) = -2$.

For ①, 

Let $A = \rho(\alpha)$, $B = \rho(\beta)$.
$\rho \in Hom(\pi, G)$.

Then $\Phi($  $)(\rho) = -tr\,AB$

$\Phi(-$  $-$  $)(\rho)$

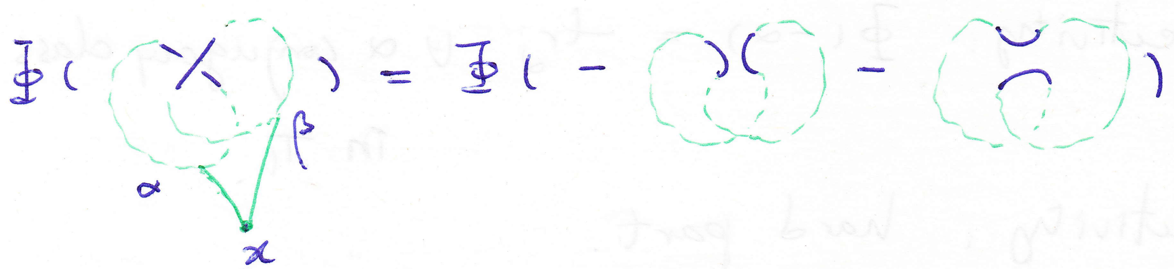
$$= -(-tr\,A)(-tr\,B) - (-tr\,AB^{-1})$$

$$= -tr\,A\,tr\,B + tr\,AB^{-1}$$

$$= -tr\,AB \quad (tr\,A\,tr\,B = tr\,AB^{-1} + tr\,AB)$$

For ②. $\Phi($  $)(\rho) = -tr\begin{pmatrix}1&0\\0&1\end{pmatrix} = -2$.

<u>algebra homomorphism</u>: it is to verify

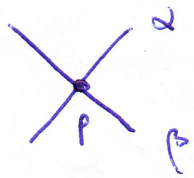


$$\Phi(\,\asymp\!\!\!\times\,)(\rho) = (-\operatorname{tr} A)(-\operatorname{tr} B) = \operatorname{tr} A \cdot \operatorname{tr} B$$

$$\Phi(-\,)(\,-\,\asymp\,) = -(-\operatorname{tr} AB) - (-\operatorname{tr} AB^{-1})$$

$$= \operatorname{tr} AB + \operatorname{tr} AB^{-1}$$

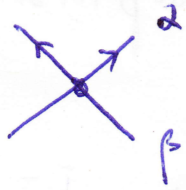
<u>Poisson alg. homo.</u>: observation



$$\varepsilon(p;\alpha.\beta)\left(\alpha_p\beta^{-} - \alpha_p\beta\right) = -)(- \asymp$$

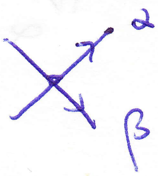
no matter how α and β are oriented

Indeed,



$$\varepsilon(p;\alpha.\beta) = 1$$

$$\alpha_p\beta^{-} = \asymp, \quad \alpha_p\beta =)(,$$



$$\varepsilon(p;\alpha.\beta) = -1.$$

$$\alpha_p\beta^{-} =)(, \quad \alpha_p\beta = \asymp.$$

Then compare the corresponding formulas.

Isomorphism

surjectivity: $\Phi(-\alpha) = tr_\alpha$, $\forall\, \alpha$ conjugacy class in π_1.

injectivity: hard part.

① Bullock: $\ker \Phi$ consists of nilpotent elements.

② P-S: Claim no nilpotent elements w/ pf.

③ C-M: independent pf using Dehn's coordinate and Foriour transform

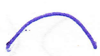
The Foriour coeff of trace function read off the Dehn ~~coeff~~ coord of the loop. $\square$

$q = 1$.

Thm 3' (Barret B., P-S, C-M) Given a spin structure sp, there is a Poisson algebra isomorphism

$$\Phi : S_1(\Sigma) \longrightarrow \mathcal{R}(\Sigma)$$

$$\alpha \longmapsto -(-1)^{sp(\alpha)}\, tr_\alpha$$

key:  $= -$ .

$sp($$) = - sp($$)$.

deformation quantization

- $\mathbb{C}[[h]]$ = ring of formal power series in h.

  h-adic topology: $\forall a \in \mathbb{C}[[h]]$,

  $\{a + h^n \mathbb{C}[[h]]\}$ form a nbhd basis.

- For a $\mathbb{C}$-vector space V,

  $V[[h]] = \mathbb{C}[[h]]$-module of f.p.s. w/ coefficients in V.

- A $\mathbb{C}[[h]]$-mod M is topologically free if $\exists$ V s.t. $M \cong V[[h]]$.

  $\cong$: isomorphism of topological $\mathbb{C}[[h]]$-mod.

- A deformation quantization of a Poisson alg $(A, \cdot, \{\,\})$ is a $\mathbb{C}[[h]]$-alg. $(A_h, \cdot)$ w/ a $\mathbb{C}$-alg. homomorphism $\theta : A_h / h A_h \to A$ so that

① As a module, A_h is topologically free

② $\forall a, b \in A$ and $\bar{a}, \bar{b} \in A_h$ so that $\theta(\bar{a}) = a$, $\theta(\bar{b}) = b$,

$$\{a, b\} = \theta\left(\frac{\bar{a}\cdot\bar{b} - \bar{b}\cdot\bar{a}}{h}\right).$$

Let $q = \pm e^{\frac{h}{4}}$, and consider

$$S_h(\Sigma) = \mathbb{C}\mathcal{I}_m[[h]] \Big/ ①②$$

Thm (Bullock - Frohman - Kania-B).

$(S_h(\Sigma), \cdot)$ is a deformation quantization of $(S_{\pm 1}(\Sigma), \cdot, \{,\})$.

Pf. For ⓐ, we need

<u>link diagram</u>: a (possibly empty) 4-valent graph in Σ w/ over-crossings and under-crossings.

<u>state</u>: a link diagram w/out crossings and trivial loops

$\mathcal{B}$ = $\mathbb{C}$-v.s generated by the isotopy classes of states.

We claim:

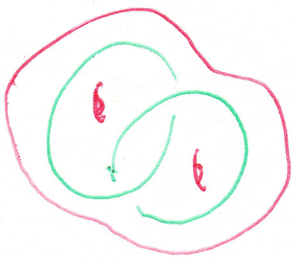
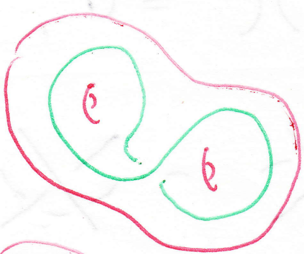
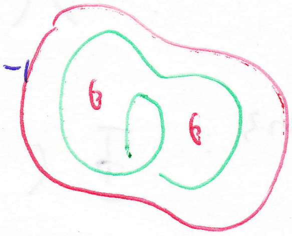
$$S_h(\Sigma) \cong \mathcal{B}[[h]].$$

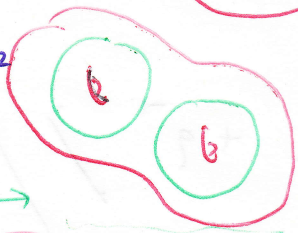
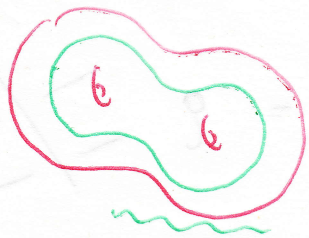
Indeed, let $\mathbb{D}$ be the $\mathbb{C}$-vector space generated by isotopy classes of link diagrams, then

$$S_h(\Sigma) \cong \mathbb{D}[[h]] \big/ ①\ ②\ m2,\ m3 .$$

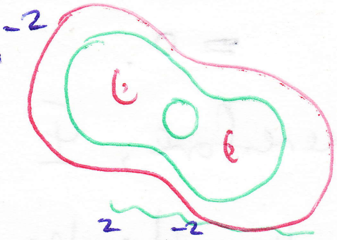
Define $\Phi : \mathbb{D}[[h]] \to \mathbb{B}[[h]]$ as follows

Step 1: resolve all the crossings using ①.

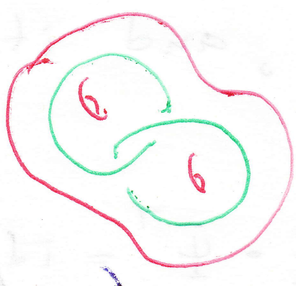
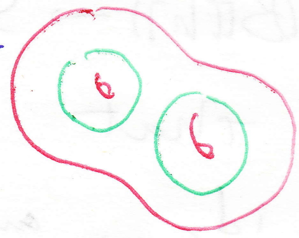
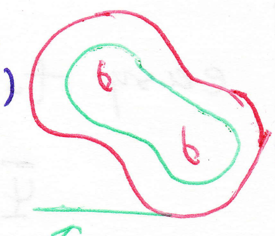
D:  $= q$  $+\, q^{-1}$ 

$= q^2$  $+$ 

smoothings

$+$  $+\, q^{-2}$ 

Step 2: send trivial loops to $-q^2-q^{-2}$ by ②.

 $= q^2$  $+\,(1-q^{-4})$ 

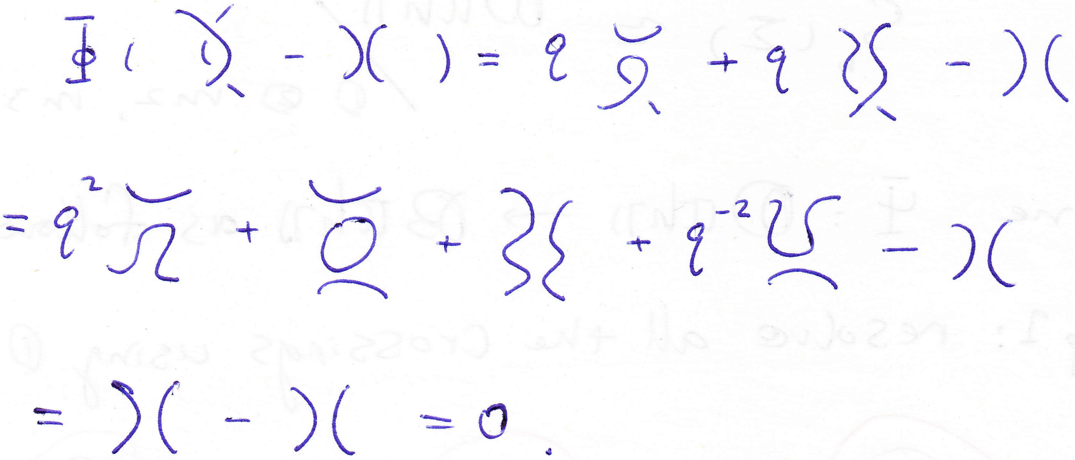
$= \Phi(D) \in \mathbb{B}[[h]]$

states

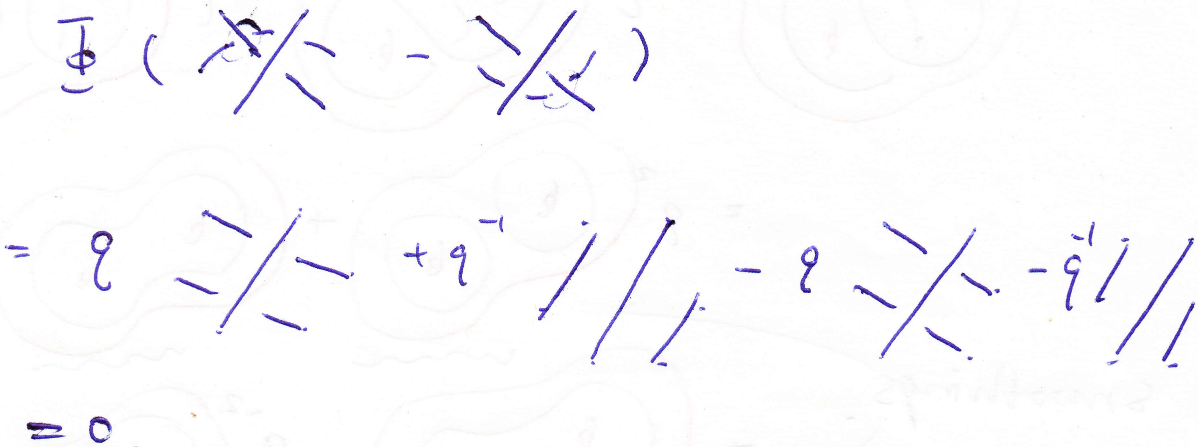
We have $\bar{\Phi}(①, ②, m_2, m_3) = 0$.

$\bar{\Phi}(①, ②) = 0$ is tautological.

m2 :



m3 :



Therefore, $\bar{\Phi}$ desents to $\bar{\Phi}: S_h(\Sigma) \to \mathcal{B}[[h]]$.

On the other hand, the inclusion $\mathcal{B} \hookrightarrow \mathcal{D}$ induces $\bar{\Phi}^{-1}: \mathcal{B}[[h]] \to S_q(\Sigma)$, and it is easy to check that

$$\bar{\Phi}^{-1} \circ \bar{\Phi} = id_{S_q(\Sigma)} \quad \text{and} \quad \bar{\Phi} \circ \bar{\Phi}^{-1} = id_{\mathcal{B}[[h]]}.$$

For b, we need

A <u>smoothing</u> S_D of a link diagram D is a link diagram obtained from D w/ all crossings resolved. e.g. —— on P_{11}.

- If D has k crossings, then there are 2^k different smoothings.
- For each S_D, let

$$\left.\begin{aligned} p(S_D) &= \#\text{ of positive resolution} \\ n(S_D) &= \#\text{ of negative resolution} \end{aligned}\right\} \text{ to get } S_D \text{ from } D.$$

Now for $\bar{\alpha}, \bar{\beta} \in \mathbb{C}\mathcal{L}_m$, let

D = link diagram of $\bar{\alpha} \cup \bar{\beta}$.

We have

$$\bar{\alpha}\cdot\bar{\beta} - \bar{\beta}\cdot\bar{\alpha} = \sum_{S_D} (q^{p(S_D)-n(S_D)} - q^{n(S_D)-p(S_D)})S$$

and

$$\{\alpha.\beta\} = \frac{1}{2}(\pm 1)^k \sum_{S_D} (p(S_D) - n(S_D))\, S,$$

which is exactly the coefficient of h in $\bar{\alpha}.\bar{\beta} - \bar{\beta}.\bar{\alpha}$.