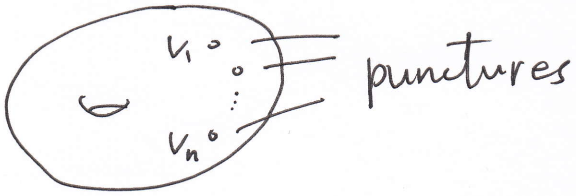
The skein algebra of arcs and links and the decorated Teichmüller space.
(Joint w/ Julien Roger)

- Σ punctured surface (smooth, oriented), $\chi(\Sigma) < 0$.

  $V = \{v_1, \ldots, v_n\}$ the set of punctures.



- Teichmüller space

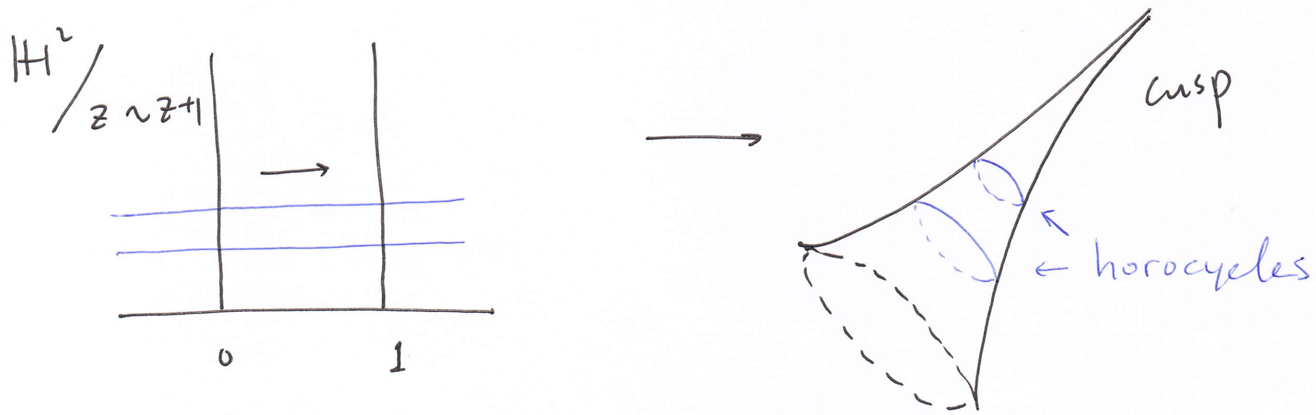
$$\mathcal{T}(\Sigma) = \{\text{complete hyperbolic metrics } d \text{ w/ finite area}\}/\sim$$

$$= \{\text{hyperbolic metric } d \text{ w/ cusp ends at } V\}/\sim$$

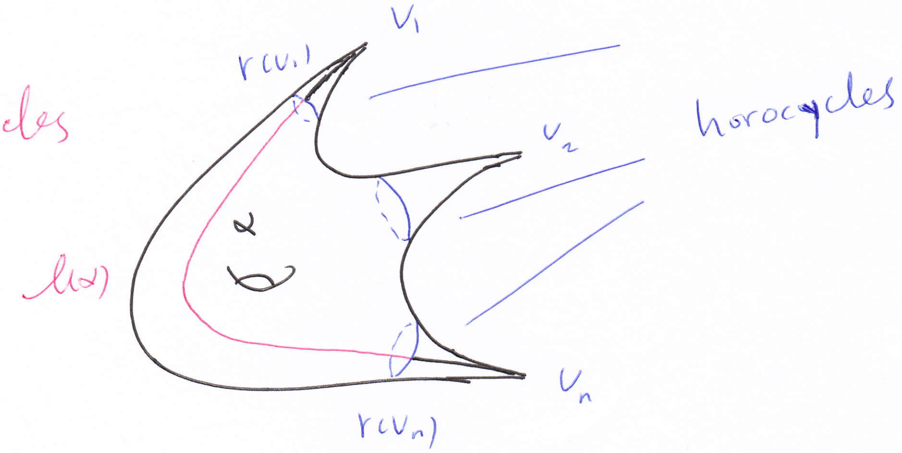
$d_1 \sim d_2$ if $\exists\, f : (\Sigma, d_1) \longrightarrow (\Sigma, d_2)$ isometry s.t.

$f \simeq id_\Sigma$.

isotopy



$l(\alpha)$ is the distance between the horocycles along α.



- a <u>decoration</u> is an assignment $r \in \mathbb{R}^V_{>0}$ of positive real numbers to the punctures.

  geometrically, they are the lengths of horocycles centered at V.

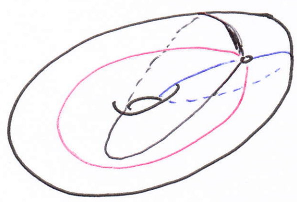
- a pair (d, r) is called a <u>decorated hyperbolic metric</u>, and the <u>decorated Teichmüller space</u>

$$\mathcal{T}^d(\Sigma) \doteq \mathcal{T}(\Sigma) \times \mathbb{R}^V_{>0}$$

is the space of decorated hyperbolic metrics.

How does this space look like? To answer this, we need

- ideal triangulation T: a triangulation of Σ so that the vertices of T are exactly the punctures V.



E = the set of edges.

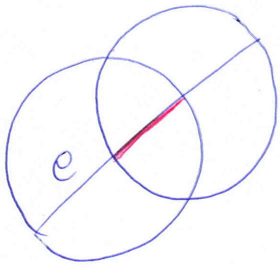
Penner's length coordinate

The map $L : \mathcal{T}^d(\Sigma) \longrightarrow \mathbb{R}^E$ defined by

$$(d, r) \longmapsto (\cdots, l(e), \cdots)$$

is a homeomorphism.

- note that, $l(e)$ could be negative.



Q1: Given a closed geodesic (or a geodesic arc) α,

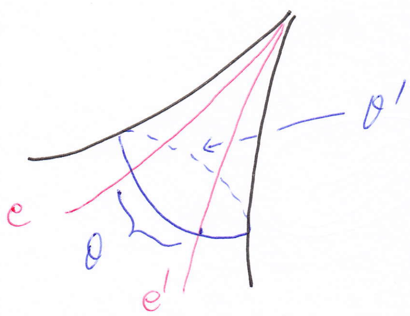
let $l_\alpha : \mathcal{T}^d(\Sigma) \longrightarrow \mathbb{R}$ be the length function by

$$(d, r) \longmapsto l_{d,r}(\alpha).$$

How do you express l_α explicitly in L?

Weil-Petersson Poisson structure (Mondello)

$$\Omega_{wp} = \frac{1}{4} \sum_{v \in V} \sum_{\substack{e \neq e' \\ e \cap e' = v}} \frac{\theta'_v - \theta_v}{r(v)} \frac{\partial}{\partial l(e)} \wedge \frac{\partial}{\partial l(e')}$$



θ : the distance from e to e' along the horocycle.

Q_2: α. β geodesics on Σ, what is

$$\Omega_{wp}(l(\alpha), l(\beta)) = ?$$

. recall Wolpert's cosine formula:

Σ closed surface, Ω_{wp} the Weil-Petersson,

α. β closed geodesics.

$$\Omega_{wp}(l(\alpha), l(\beta)) = \frac{1}{2} \sum_{p \in \alpha \cap \beta} \cos \theta_p$$

Q_3: How to quantize $\mathcal{T}^d(\Sigma)$?

recall last two lectures

$$\begin{tikzcd} S_q(\Sigma) \arrow[dr, "\text{quantization}"] & \\ S_{\pm 1}(\Sigma) \arrow[u, "\text{quantization}"] \arrow[r, phantom, "\cong"] & R(\Sigma) \end{tikzcd}$$

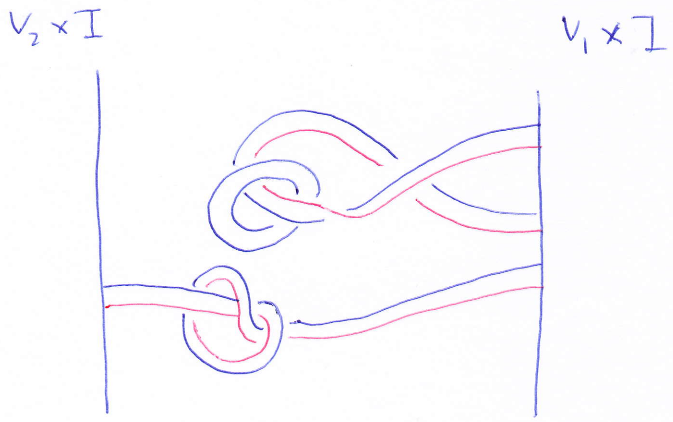
the skein algebras.

ring of regular functions over $\mathcal{X}(\Sigma)$, the $SL(2,\mathbb{C})$ character variety.

$\mathcal{T}(\Sigma) \subset \mathcal{X}(\Sigma)$

as the top dimensional connected component.

- A <u>generalized framed link</u> in $\Sigma \times I$:



- $\mathcal{I} = \{$ isotopy class of generalized framed links $\}$
- $\mathbb{C}[[h]]$ = ring of formal power series in h.

<u>the skein algebra of arcs and links</u> :

$$\mathcal{AS}_h(\Sigma) = \mathbb{C}[\mathcal{I}, V^{\pm}][[h]] \Big/ ①②③④$$

① Kauffman bracket skein relation. $q = e^{\frac{h}{4}} \in \mathbb{C}[[h]]$.

 $= q$  $+ q^{-1}$  .

could be parts of either arcs or links.

② Framing relation :

 $= -q^2 - q^{-2}$.

③ Puncture skein relation :

 $= \frac{1}{v}\left(q^{\frac{1}{2}}\right.$  $+ q^{-\frac{1}{2}}$  $\left.\right)$.

④ Puncture relation :

 $= q + q^{-1}$.

- multiplication $\cdot$: $\alpha, \beta \in \mathcal{L}$,

$$\alpha \cdot \beta = \text{"stacking } \alpha \text{ above } \beta \text{ along } I\text{."}$$

- $\hbar = 0$, $(\mathcal{A}S_0(\Sigma), \cdot)$ is commutative:

- $\{,\} : \mathcal{A}S_0(\Sigma) \times \mathcal{A}S_0(\Sigma) \longrightarrow \mathcal{A}S_0(\Sigma)$

$$\{\alpha, \beta\} = \frac{1}{2} \sum_{\substack{p \in \alpha \cap \beta \\ \text{on } \Sigma}} (\alpha_p \beta^+ - \alpha_p \beta^-) + \frac{1}{4} \sum_{\substack{v \in \alpha \cap \beta \\ \text{at } V}} \frac{1}{v} (\alpha_v \beta^+ - \alpha_v \beta^-)$$

Thm 1 (Roger - Y.)

1) $(\mathcal{A}S_\hbar(\Sigma), \cdot)$ is a well defined associative $\mathbb{C}[[\hbar]]$-algebra.

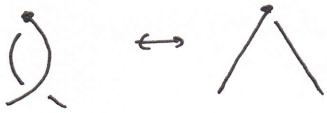
2) $(\mathcal{A}S_0(\Sigma), \cdot, \{,\})$ is a well defined Poisson algebra.

3) $(\mathcal{A}S_\hbar(\Sigma), \cdot)$ is a quantization of $(\mathcal{A}S_0(\Sigma), \cdot, \{,\})$, i.e.,

$$\{\alpha, \beta\} = \frac{\alpha \cdot \beta - \beta \cdot \alpha}{\hbar} \pmod{\hbar}.$$

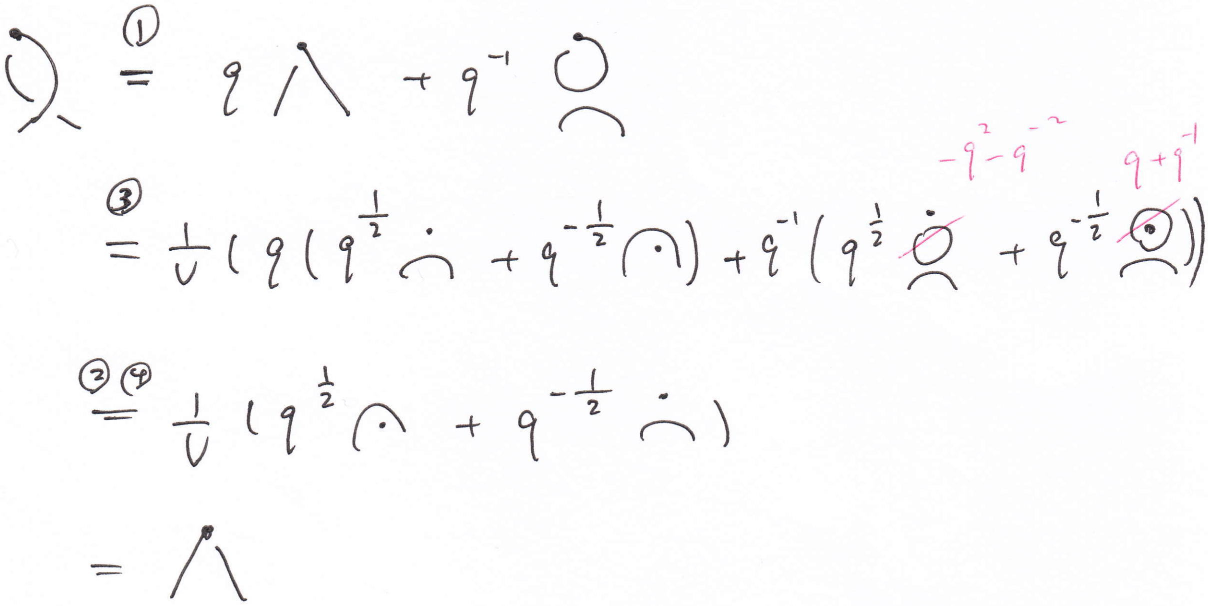
pf: to work with link diagrams, we need to verify the invariance under

R2: , R3:  and

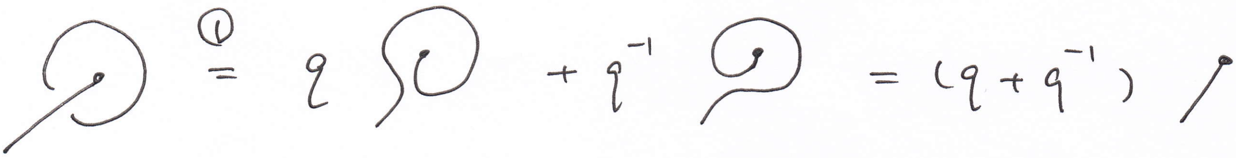
R2': 

R2 and R3 are "喜闻乐见".

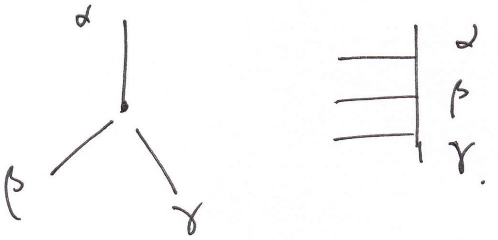
R2':

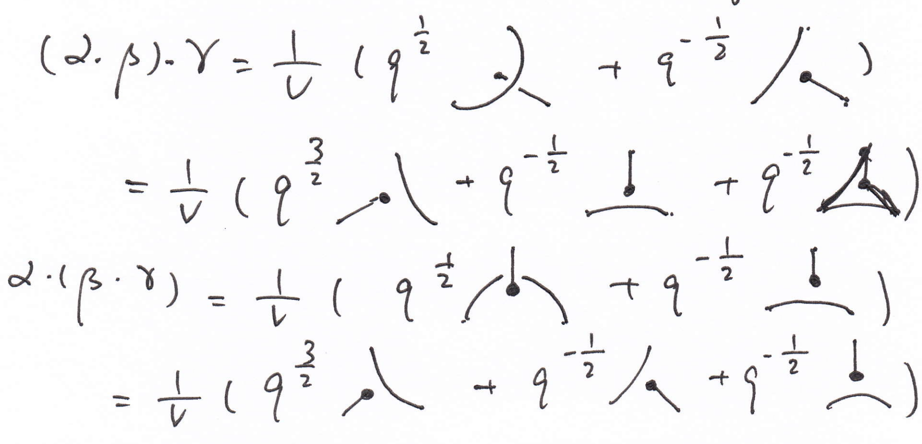


why puncture relation ④ ?



why puncture skein relation ③ ? For associativity.



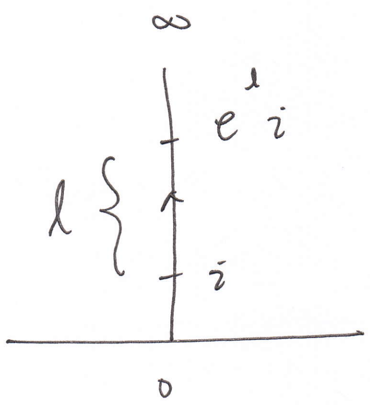


Back to hyperbolic geometry

In the closed ~~et~~ case: $\phi : S_{-1}(\Sigma) \longrightarrow \mathcal{R}(\Sigma)$

$$\alpha \longmapsto -\mathrm{tr}_\alpha .$$

For $PSL(2,\mathbb{R})$, there is no trace, but let



$\phi : z \longmapsto e^l \cdot z$.

in matrix, $\phi = \begin{pmatrix} e^{\frac{l}{2}} & 0 \\ 0 & e^{-\frac{l}{2}} \end{pmatrix}$, and

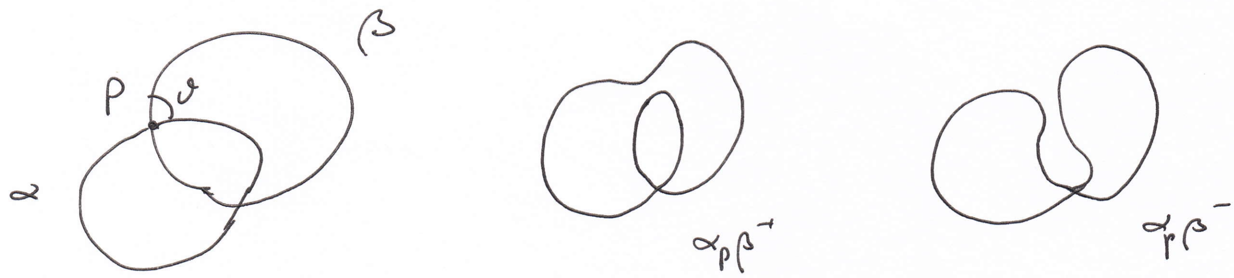
$$\boxed{\mathrm{tr}\,\phi = 2\cosh\frac{l}{2}}$$

Basic idea: we can replace trace by $\underline{2\cosh\frac{l}{2}}$.

Q: what's the counter-part for arcs? $\underline{e^{\frac{l}{2}}}$.

<u>lengths identitys</u>:

(1) α, β closed geodesics.

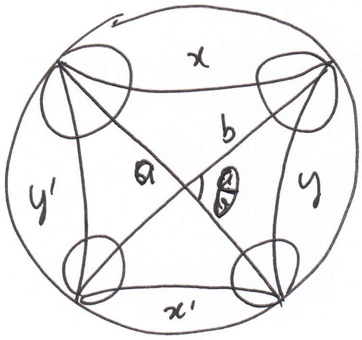


$a = l(\alpha)$, $b = l(\beta)$, $x = l(\alpha_p\beta^{-1})$ and $y = l(\alpha_p\beta^{-})$.

$$\begin{cases} \cosh\frac{x}{2} + \cosh\frac{y}{2} = 2\cosh\frac{a}{2}\cosh\frac{b}{2} \\ \cosh\frac{x}{2} - \cosh\frac{y}{2} = 2\sinh\frac{a}{2}\sinh\frac{b}{2}\cos\theta \end{cases}$$

(2) Penner's Ptolemy relation

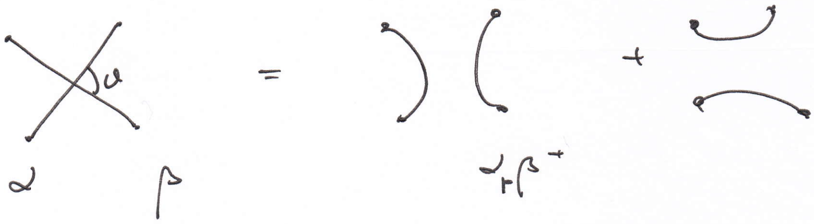
$\mathbb{H}^2$



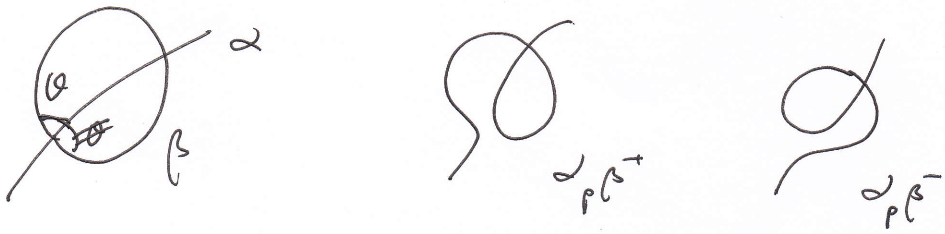
$$\begin{cases} e^{\frac{x}{2}}\cdot e^{\frac{x'}{2}} + e^{\frac{y}{2}}\cdot e^{\frac{y'}{2}} = e^{\frac{a}{2}}\cdot e^{\frac{b}{2}} \\ e^{\frac{x}{2}}\cdot e^{\frac{x'}{2}} - e^{\frac{y}{2}}\cdot e^{\frac{y'}{2}} = e^{\frac{a}{2}}\cdot e^{\frac{b}{2}}\cos\theta \end{cases}$$

It is a skein relation!!!

α, β geodesic arcs.



(3) α geodesic arc, β closed geodesic.



$$\begin{cases} e^{\frac{x}{2}} + e^{\frac{y}{2}} = 2e^{\frac{a}{2}}\cosh\frac{b}{2} \\ e^{\frac{x}{2}} - e^{\frac{y}{2}} = 2e^{\frac{a}{2}}\sinh\frac{b}{2}\cos\theta \end{cases}$$

(4) (5) (6) (7)

Let Σ be a punctured surface, and let α be a curve on Σ.

Fix a $(d,r) \in \mathcal{T}^d(\Sigma)$, and let $a = l(\alpha)$ in (d,r).

<u>Def</u>:

$$\lambda(\alpha) = \begin{cases} 2\cosh\frac{a}{2}, & \alpha \text{ loop} \\ e^{\frac{a}{2}}, & \alpha \text{ arc}. \end{cases}$$

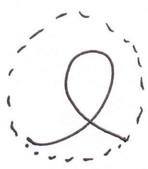
<u>Def</u>:

$$\Phi : \mathcal{S}S_0(\Sigma) \longrightarrow C^\infty(\mathcal{T}^d(\Sigma)).$$

$$v \longmapsto r(v)$$

$$\alpha \longmapsto (-1)^{c(\alpha)}\lambda(\alpha).$$

- $c(\alpha)$ = # of <u>curls</u> α contains.



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e.g. $c(\text{a geodesic}) = 0$, $c(\bigcirc) = 1$.

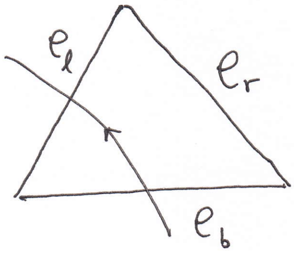
Thm 2 (Roger – Y.) Φ is a well defined Poisson algebra homomorphism w.r.t. $\{,\}$ and ω_{wp}.

Pf: (5) – (7) $\Rightarrow$ well definition.

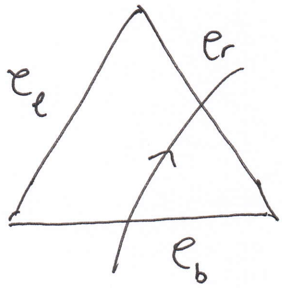
1st $\frac{1}{2}$ of (1) – (4) $\Rightarrow$ algebra homomorphism.

2nd $\frac{1}{2}$ of (1) – (4) $\Rightarrow$ Poisson algebra homomorphism

Cor 1: (Answer to Q1). Given an orientation of α,



left turn: $M_l = \begin{pmatrix} \lambda(e_b) & \lambda(e_r) \\ 0 & \lambda(e_l) \end{pmatrix}$



right turn: $M_r = \begin{pmatrix} \lambda(e_r) & 0 \\ \lambda(e_l) & \lambda(e_b) \end{pmatrix}$

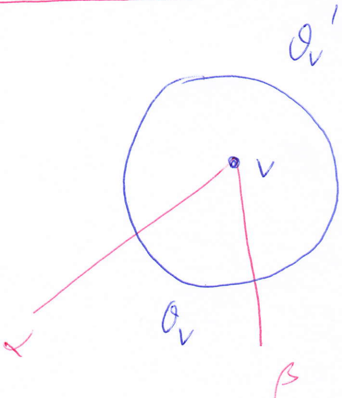
Let $\Delta_{i_1}, \ldots, \Delta_{i_n}$ be the triangles that α goes through, and let $e_{j_1}, \ldots, e_{j_n}$ be the edges that α intersects.

Then

$$\boxed{2\cosh \frac{l(\alpha)}{2} = \frac{\operatorname{tr}(M_{i_1} \cdots M_{i_n})}{\lambda(e_{j_1}) \cdots \lambda(e_{j_n})}}$$

Cor 2: (generalized Wolpert's cosine formula).

$$\boxed{\Omega_{WP}(a,b) = \frac{1}{2} \sum_p \cos\theta_p + \frac{1}{4} \sum_v \frac{\theta_v' - \theta_v}{r(v)}}$$



$r(v) = \theta_v + \theta_v'$