Quantum Invariants: A skein theoretical approach ①

Lecture 1: Introduction

History:

- Jones (mid 80's): Jones polynomials. $J(K)$

  defined using rep'n theory of operator algebras.

(2 years later, Kauffman, skein theory)

- Witten (89): Witten's invariants $W_n(M) \in \mathbb{C}$, $n \in \mathbb{N}$

  reinterpreted Jones and defined

  using Quantum Field Theory (Chern-Simons)

s.t.
1. $W_n(M \# N) = W_n(M) \cdot W_n(N)$
2. $W_n(-M) = \overline{W_n(M)}$
3. $W_n(S^3) = 1 \qquad \forall n \in \mathbb{N}$.

More generally, $W_n(M, L)$, where $L \subset M$ link.

(2)

- Reshetikhin – Turaev (90, 91) mathematical realization of Witten's invariants.

– Colored Jones Polynomials $J_n(K)$, $n \in \mathbb{N}$

– Reshetikhin – Turaev invariants $RT_n(M) \in \mathbb{C}$, $n \in \mathbb{N}$

oriented, closed 3-manifold.

using surgery along links

+ rep'n theory of quantum groups.

(later, Lickorish, Blanchet – Hambegger – Masbaum – Vogel

skein theory)

General method of defining an invariant

- presentation of M, eg. triangulation, surgery, Heegaard splitting.
- put algebraic ingredients to cook up a quantity.
- show invariance under certain moves eg.

Pachner moves for triangulations

Kirby moves for surgery diagrams

Elementary moves for Heegaard splittings.

- Turaev–Viro (92)

– Turaev–Viro invariants $TV_n(M) \in \mathbb{R}$, $n \in \mathbb{N}$

could be non-orientable, w/ or w/out boundary.

Using triangulations + rep'n of quantum gps.

(later, Kauffman–Lins, skein theory.)

- Thm (Turaev, Walker, Roberts)

skein theory.

$$TV_n(M) = |RT_n(M)|^2.$$

Heegaard splitting of M

triangulation of M

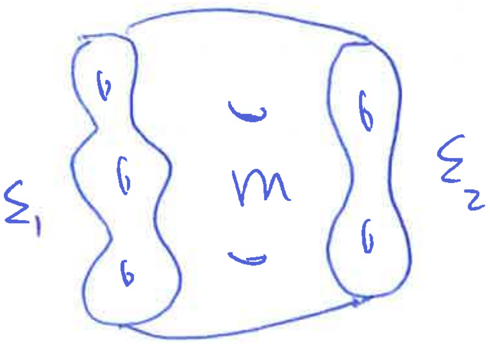
Surgery diagram of $M \# (-M)$.

$$\Rightarrow TV_n(M) = RT_n(M \# (-M)) = RT_n(M) \cdot \overline{RT_n(M)}$$

1st half: invariants

2nd half: Topological Quantum Field Theory (TQFT) ④

"invariant" of a cobordism.



$\Rightarrow$ $L_M : V(\Sigma_1) \to V(\Sigma_2)$.

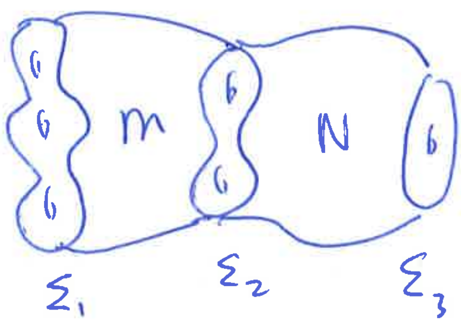
linear map, vector spaces.

s.t.

① : $L_{\Sigma \times I} = id_{V(\Sigma)}$

② : $L_{M \cup_{\Sigma_2} N} = L_N \circ L_M$

functor form the category of surfaces and cobordisms Cob^2 to the category of vector spaces and linear maps Vec.



③ $V(\phi) =$ base field $\mathbb{k}$ $(\mathbb{R}, \mathbb{C})$

④ $V(\Sigma_1 \amalg \Sigma_2) \cong V(\Sigma_1) \otimes V(\Sigma_2)$

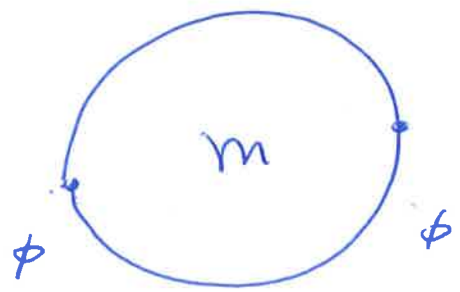
⑤ $V(-\Sigma) \cong V(\Sigma)^*$

⋮

recall properties of Witten's invariants

TQFT's give

(1) invariants of closed 3-mfds



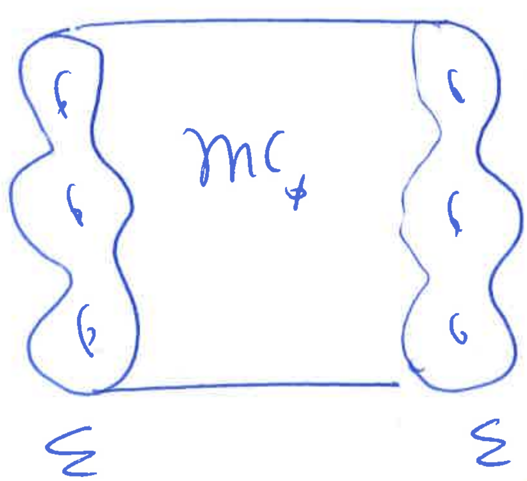
$\Rightarrow L_m : \mathbb{C} \to \mathbb{C}$

$\Rightarrow \underline{I(m)} \doteq L_m(1) \in \mathbb{C}$.

invariant

(2) rep'ns of mapping class group $\mathrm{Mod}(\Sigma)$

For $\phi : \Sigma \to \Sigma$ self-homeomorphism of Σ, consider the mapping cylinder mc_ϕ, which



could be considered as a cobordism between Σ and itself.

Then $\Rightarrow L_{mc_\phi} : V(\Sigma) \to V(\Sigma)$

$\Rightarrow \rho : \mathrm{Mod}(\Sigma) \longrightarrow \mathrm{End}(V(\Sigma))$

$[\phi] \longmapsto L_{mc_\phi}$

- Turaev-Viro, Reshetikhin-Turaev TQFT's

(BHMV, 95, skein theory)

$\Rightarrow$ TV-, RT- representations of $\mathrm{Mod}(\Sigma)$.

Thm (Freedman - Walker - Wang 王, 2002)

TV-, RT-rep'ns are asymptotically faithful,

i.e., $\forall \phi \in \mathrm{Mod}(\Sigma)$, $\exists\, n$ and $v \in V_n(\Sigma)$ s.t

$$\rho_n(\phi)(v) \neq v.$$

Open question: Is $\mathrm{Mod}(\Sigma)$ linear?

True for Σ_2, $\Sigma_{0,n}$. (Bigelow).

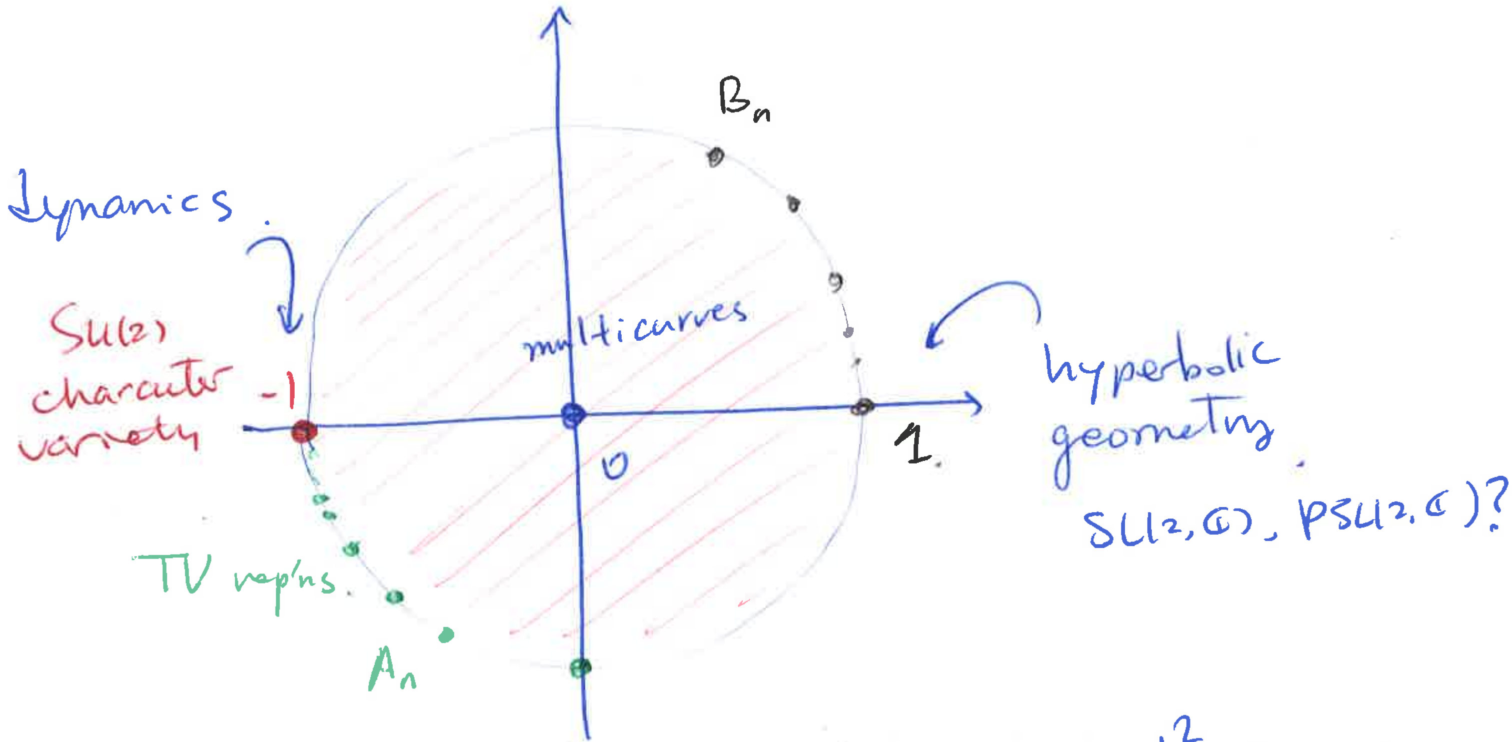
- Costantino - Martelli (2014)

– Analytic family of $\rho_A : \mathrm{Mod}(\Sigma) \to \mathrm{End}(\mathcal{H}_A)$, $A \in \overline{\mathbb{D}}$.

(Hilbert space, separable)

s.t.

- $A = 0$, $\mathcal{H}_A = \ell^2(\text{multicurves on } \Sigma)$
- $A = -e^{\frac{\pi i}{4n}}$, $4n$-th root of 1, $\mathcal{H}_A =$ Turaev-Viro vector space.
- $A = -1$, $\mathcal{H}_A = L^2(SU(2)\text{-character variety } X_{SU(2)}(\Sigma))$.



It's well known that $\mathrm{Mod}(\Sigma)$ action on $L^2(X(\Sigma))$ is faithful

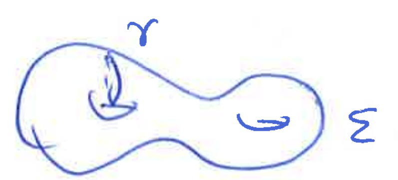
$\Rightarrow$ Thm of Freedman–Walker–Wang.

Witten: $\mathrm{Vol}(X_{SU(2)}(\Sigma_g)) = \zeta(2g-2) = \sum_{n=1}^{\infty} \frac{1}{n^{2g-2}}$

$$\| \quad \int_{X_{SU(2)}(\Sigma_g)} 1 \, d\mu$$

Atiyah–Bott, Goldman measure.

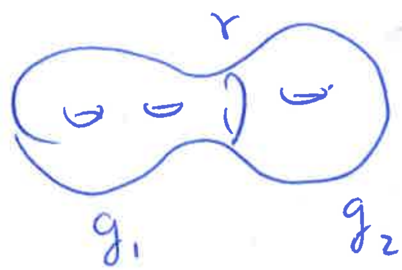
Q: $\gamma \subset \Sigma$ simple closed curve.



$$\int_{X_{SU(2)}(\Sigma_g)} tr_\gamma \, d\mu = ?$$

Answer (Frohman - Kania-Bartosynska, 2004).

$$= \begin{cases} 0, & \gamma \text{ non-separating} \\ -\sum_{n=1}^{\infty} \left(\dfrac{1}{n^{2g_1-1}(n+1)^{2g_2-1}} + \dfrac{1}{n^{2g_2-1}(n+1)^{2g_1-1}} \right), & \text{if} \end{cases}$$



Using quantum techniques!!!

Conj (Chen - Y., 2015) M hyperbolic, ~~then~~ $B_n = A_n^2 = e^{\frac{\pi i}{2n}}$,

then $$\lim_{n\to+\infty} \frac{2\pi}{n} \ln \left(TV_n(M, B_n) \right) = \mathrm{Vol}_{\mathbb{H}^3}(M).$$

Q. Why Quantum?

1. It's related and may shed light on dynamics and hyperbolic geometry.
2. Interesting on its own.

Q. Why skein?

1. Easy to learn.
2. Given new things, e.g. RT-TQFT's, CM-family of rep'ns