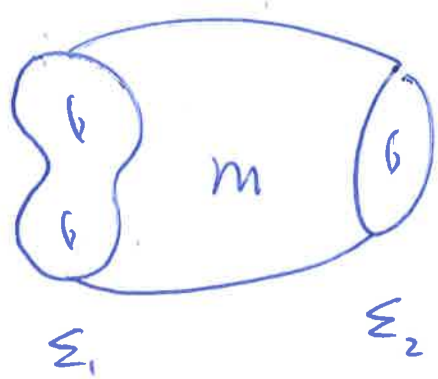
Lecture 11: Turaev-Viro TQFT

Category Cob_2:

Objects: closed orientable surfaces

- not necessarily connected, oriented.
- not homeomorphism classes of surfaces.

Morphisms from Σ_1 to Σ_2: equivalence classes of cobordisms from Σ_1 to Σ_2.

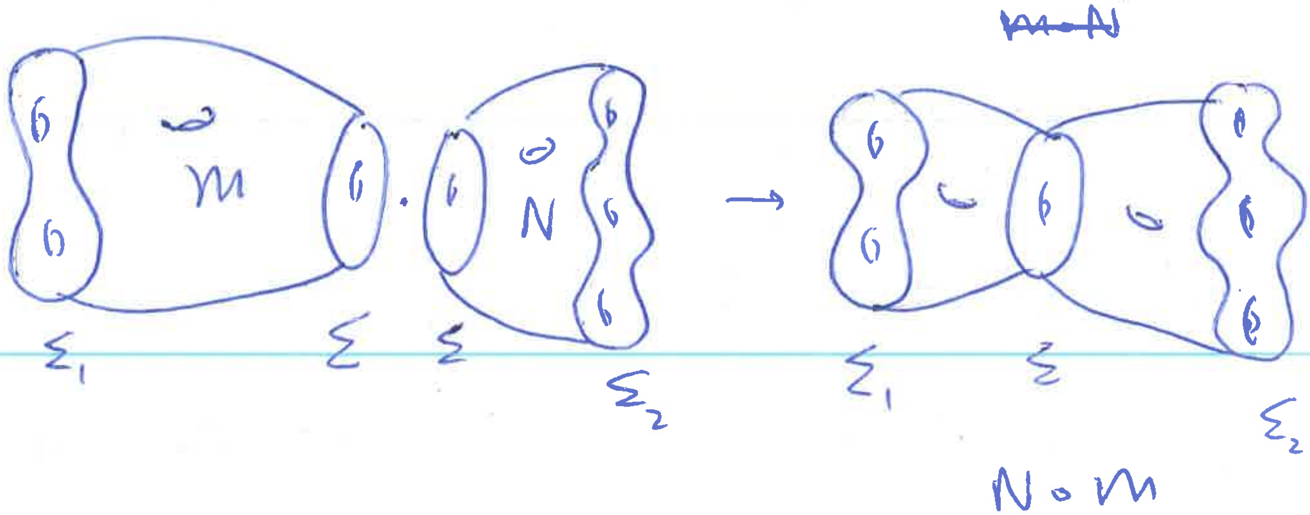


A cobordism from Σ_1 to Σ_2 is a compact 3-mfd M s.t. $\partial M = \Sigma_1 \sqcup \Sigma_2$.

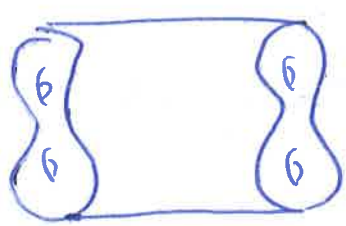
Two cobordisms M_1 and M_2 are equivalent if they are homeomorphic rel. boundary, ie, the homeomorphism restricts to identities on Σ_1 and Σ_2.

Rm. this is to make the composition associative.

<u>Composition</u>: glue cobordisms together along Σ by id_Σ.



<u>identity</u>: $id_\Sigma = \Sigma \times [0,1]$.



Cob_2 has extra structures: disjoint union $\amalg$
empty obj $\emptyset$.

Rm: sometimes, consider cobordism w/ structures.

Category $Vect_{\mathbb{R}}$: isomorphism classes of finite dimensional $\mathbb{R}$-vector spaces and linear maps.

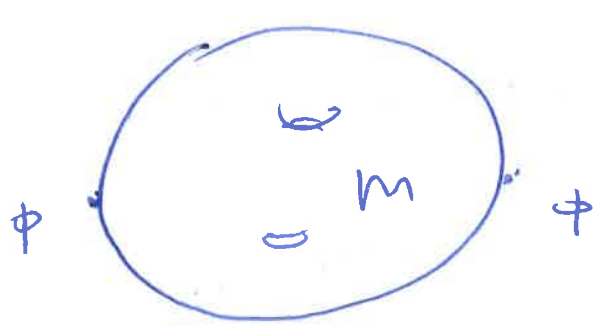
Def: A (2+1)-topological quantum field theory is a functor $F: \mathcal{C}ob_2 \to Vect_{\mathbb{R}}$ s.t.

(1) $F(\phi) \cong \mathbb{R}$ and

(2) $F(\Sigma_1 \amalg \Sigma_2) \cong F(\Sigma_1) \otimes F(\Sigma_2)$.

Rm: If M is a closed 3-mfd, then F defines an invariant $I(M)$ of M by

$F(M)(1) \in \mathbb{R}$. $\quad M : \phi \to \phi$

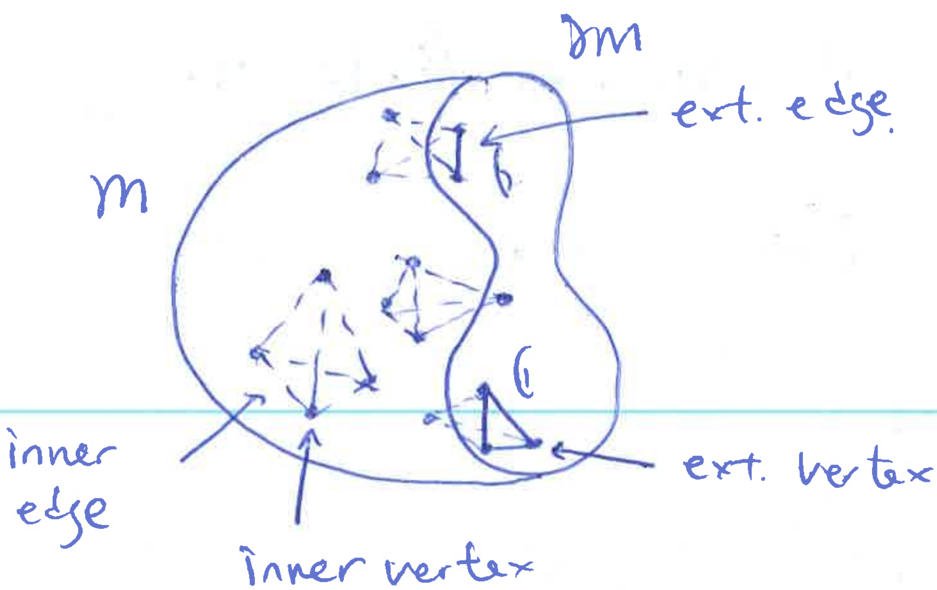


$$\begin{array}{ccc} F(M): F(\phi) & \longrightarrow & F(\phi) \\ \wr\wr & & \wr\wr \\ \mathbb{R} & & \mathbb{R} \\ 1 & \longmapsto & I(M) \end{array}$$

To define TV - TQFT, we need

Relative TV - invariants.

Let M be a compact triangulated 3-mfd w/ $\partial M \neq \emptyset$



Let V_{in}, V_{ex}, E_{in}, E_{ex}, T be the sets of inner vertices, exterior vertices, inner edges, exterior edges and tetrahedra.

- $r \geq 3$, $I_r = \{0, 1, \cdots, r-2\}$

A coloring $\alpha : E_{ex} \to I_r$ is r-adm if $\forall$ face (triangle with sides a, b, c), (a,b,c) is r-admissible.

For such α, denote by $\mathcal{A}_r(\alpha) \doteq \mathcal{A}_r(M, \mathcal{T}, \alpha)$ the set of r-adm colorings of $\mathcal{T}$ that restricts to α on ∂M.

<u>Def</u>: The relative TV-invariant of M is

$$TV_r(M, \alpha) = \eta^{-|V_{in}| - \frac{1}{2}|V_{ex}|} \prod_{e \in E_{ex}} |e|_\alpha^{\frac{1}{2}} \sum_{c \in \mathcal{A}_r(\alpha)} \prod_{e \in E_{in}} |e|_c \prod_{\sigma \in T} \|\sigma\|_c$$

If $\mathcal{A}_r(\alpha) = \phi$, then $TV_r(M, \alpha) = 0$ as the sum of empty set of summands.

Thm 1: Any $\mathcal{T}_1$ and $\mathcal{T}_2$ of M that coincide on ∂M give the same $TV_r(M, \alpha)$, for any α.

For triangulated closed surface Σ, let

$$C_r(\Sigma) = \mathrm{Span}_{\mathbb{R}} \{ r\text{-adm colorings of } \Sigma \}$$

$$= \mathbb{R}(\mathcal{A}_r(\Sigma)).$$

Let $\langle , \rangle_\Sigma : C_r(\Sigma) \times C_r(\Sigma) \to \mathbb{R}$ be inner product

s.t. the admissible colorings are orthonormal.

- $C_r(\phi) = \mathbb{R}$, since $\exists!$ coloring $\phi \to I_r$.

- For cobordism $M : \Sigma_1 \to \Sigma_2$ between triangulated Σ_1 and Σ_2, choose a triangulation $\mathcal{T}$ of M extending that of Σ_1 and Σ_2.

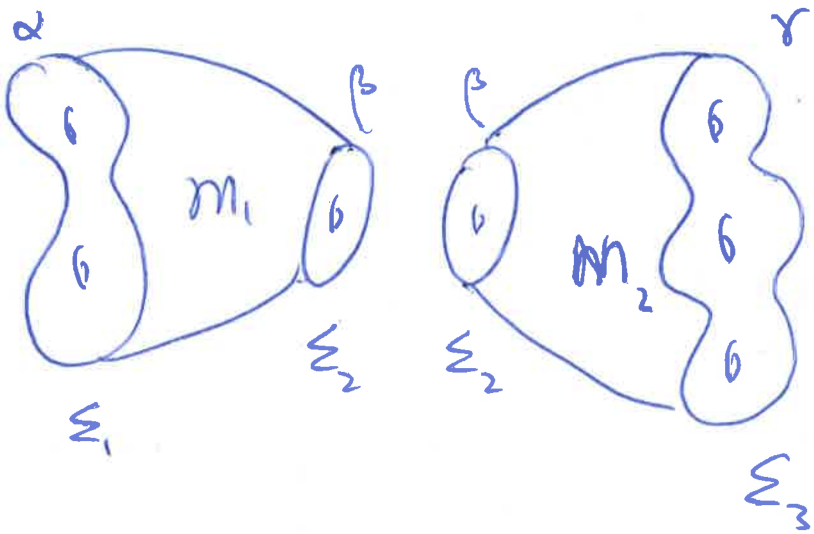
Define $\Phi_M : C_r(\Sigma_1) \to C_r(\Sigma_2)$ by

$$\boxed{\Phi_M(\alpha) = \sum_{\beta \in \mathcal{A}_r(\Sigma_2)} TV_r(M, \alpha \cup \beta) \cdot \beta}, \quad \alpha \in \mathcal{A}_r(\Sigma_1).$$

Cor of Thm 1: Φ_M does not depend on $\mathcal{T}$.

Prop 1: $\Phi_{M_2 \circ M_1} = \Phi_{M_2} \circ \Phi_{M_1}$

Pf:



Let $\alpha \in \mathcal{A}_r(\Sigma_1)$, $\gamma \in \mathcal{A}_r(\Sigma_3)$.

Then

$$TV_r(M_1 \cup_{\Sigma_2} M_2,\ \alpha \cup \gamma)$$

$$= \sum_{\beta \in \mathcal{A}_r(\Sigma_2)} TV_r(M_1, \alpha \cup \beta) \cdot TV_r(M_2, \beta \cup \gamma).$$

□

Consider $id_\Sigma = \Sigma \times I : \Sigma \longrightarrow \Sigma$.

In general $\Phi_{id_\Sigma} : C_r(\Sigma) \to C_r(\Sigma)$ is not $id_{C_r(\Sigma)}$,

indeed, it is not injective.

We need "abstract nonsense".

let $V_r(\Sigma) \doteq C_r(\Sigma) / \ker \Phi_{id_\Sigma}$ $\left(= \text{coimage of } \Phi_{id_\Sigma} \right)$.

For $m : \Sigma_1 \longrightarrow \Sigma_2$, the linear map $\Phi_m : C_r(\Sigma_1) \to C_r(\Sigma_2)$

induces $\Psi_m : V_r(\Sigma_1) \to V_r(\Sigma_2)$.

Indeed, we need to show

$$\Phi_m (\ker_{id_{\Sigma_1}}) \subset \ker_{id_{\Sigma_2}}$$

Since $m \circ id_{\Sigma_1} = m$, $\Phi_m \circ \Phi_{id_{\Sigma_1}} = \Phi_m$.

$\Rightarrow$ $\ker \Phi_{id_{\Sigma_1}} \subset \ker \Phi_m$

$\Rightarrow$ $\Phi_m (\ker \Phi_{id_{\Sigma_1}}) = 0 \in \ker \Phi_{id_{\Sigma_2}}$.

Prop 2.

(1) $\overline{\Phi}_m$ is injective,

(2) $\overline{\Phi}_{m_2 \circ m_1} = \overline{\Phi}_{m_2} \circ \overline{\Phi}_{m_1}$, and

(3) $\overline{\Phi}_{id_\Sigma} = id_{V(\Sigma)}$.

Pf: (1) By definition. (2) By Prop 1.

(3) $id_\Sigma \circ id_\Sigma = id_\Sigma \overset{(2)}{\Rightarrow}$

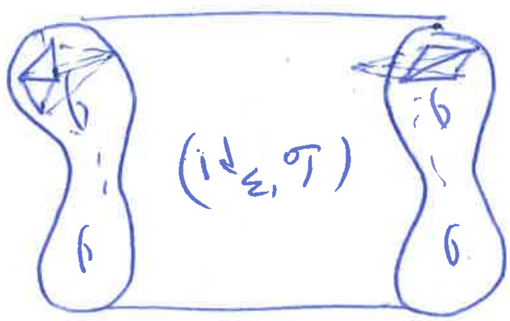
$$\overline{\Phi}_{id_\Sigma} \circ \overline{\Phi}_{id_\Sigma} = \overline{\Phi}_{id_\Sigma} = \overline{\Phi}_{id_\Sigma} \circ id_{V(\Sigma)}.$$

Since by (1) $\overline{\Phi}_{id_\Sigma}$ is injective, $\overline{\Phi}_{id_\Sigma} = id_{V(\Sigma)}$. ∎

Prop 3: $V(\Sigma)$ does not depend on the triangulation of Σ in the sense that there is a natural isomorphism between $V(\Sigma, \mathcal{T}_1)$ and $V(\Sigma, \mathcal{T}_2)$.

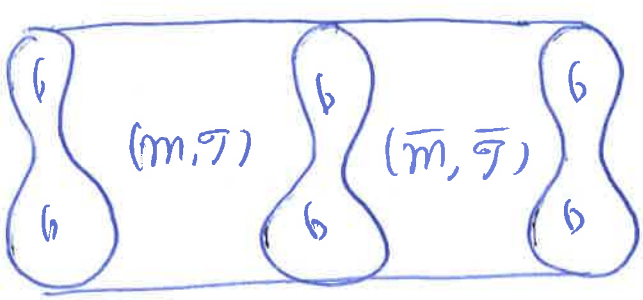
pf: Consider $(id_\Sigma, \mathcal{T}) : (\Sigma, \mathcal{T}_1) \to (\Sigma, \mathcal{T}_2)$

where $\mathcal{T}$ is an triangulation of $\Sigma \times I$ extending $\mathcal{T}_1$ and $\mathcal{T}_2$.



$(\Sigma, \mathcal{T}_1)$ $(\Sigma, \mathcal{T}_2)$

This induces $\Phi_m : V(\Sigma, \mathcal{T}_1) \to V(\Sigma, \mathcal{T}_2)$.



$(\Sigma, \mathcal{T}_1)$ $(\Sigma, \mathcal{T}_2)$ $(\Sigma, \mathcal{T}_1)$.

By Prop 1,

$\Phi_m \circ \Phi_{\bar{m}} = \Phi_{id_\Sigma} = id_{V(\Sigma)}$

$\Rightarrow \Phi_m$ is injective and $\Phi_{\bar{m}}$ is surjective $\Rightarrow \Phi_m$ is surjective $\square$.

Thm (TV). $F : \mathcal{C}ob_2 \to Vect_{\mathbb{R}}$ defines a TQFT.

$\Sigma \longmapsto V(\Sigma)$

$m \longmapsto \Phi_m$

Representation of mapping class group

Let $\phi : \Sigma \to \Sigma$ be a self-homeomorphism.

Fix a triangulation $\mathcal{T}$ of Σ. Then $\phi(\mathcal{T})$ is a triangulation of target Σ, and for $\alpha \in \mathcal{A}_r(\Sigma, \mathcal{T})$, ϕ induces $\phi(\alpha) \in \mathcal{A}_r(\Sigma, \phi(\mathcal{T}))$.

Define $\phi_{\#} : C_r(\Sigma) \to C_r(\Sigma)$

$$\alpha \longmapsto \sum_{\beta \in \mathcal{A}_r(\Sigma, \mathcal{T})} TV(\Sigma \times I, \beta \cup \phi(\alpha)) \cdot \beta$$

Prop1 $\Rightarrow$ $(\psi \circ \phi)_{\#} = \psi_{\#} \circ \phi_{\#}$.

By the same argument, $\phi_{\#}$ induces

$$\phi_* : V_r(\Sigma) \to V_r(\Sigma).$$

Thm: (i) $(id_\Sigma)_* = \Phi_{id_\Sigma} = id_{V(\Sigma)}$.

(ii) $(\psi \circ \phi)_* = \psi_* \circ \phi_*$

(iii) $(\phi^{-1})_* \circ \phi_* = (\phi^{-1} \circ \phi)_* = id_{V(\Sigma)}$

$\Rightarrow$ If ϕ is isotopic to ψ, then $\phi_{\#} = \psi_{\#}$, and hence $\phi_* = \psi_*$.

Indeed, we have

$$TV_r(\Sigma\times I,\ \beta\cup\phi(\alpha)) = TV_r(\Sigma\times I,\ \beta\cup\psi(\alpha)),$$

since $\exists$ self-homeomorphism of $\Sigma\times I$ that is identity on $\Sigma\times\{0\}$ and sents $\phi(\alpha)$ to $\psi(\alpha)$.

Thm 2 (TV). $\rho_r : \mathcal{M}od(\Sigma) \longrightarrow End(V_r(\Sigma))$

$$\phi \longmapsto \phi_*$$

defines a representation of $\mathcal{M}od(\Sigma)$.