last time.

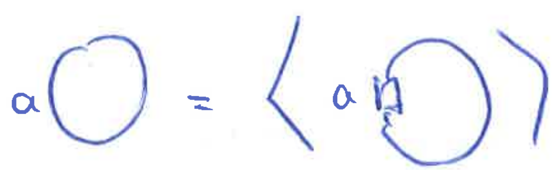
For $r \geq 3$, let $A_r = e^{\frac{\pi i}{2r}}$ and $I_r = \{0, 1, \dots, r-2\}$

$\cdot\ (a,b,c) \in I_r^3$ is ~~ad~~ r-admissible if

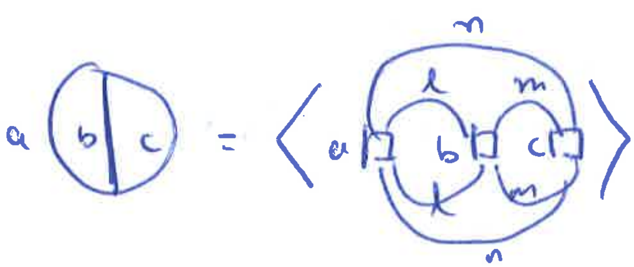
(i) $a+b \geq c$, $b+c \geq a$, $c+a \geq b$.

(ii) $a+b+c \leq 2(r-2)$,

(iii) $a+b+c$ is even.

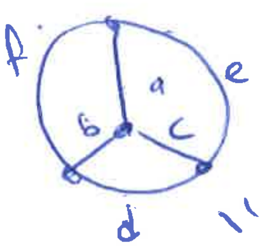


$= (-1)^a [n+1]$



$l = \frac{a+b-c}{2}$, $m = \frac{b+c-a}{2}$, $n = \frac{a+c-b}{2}$.

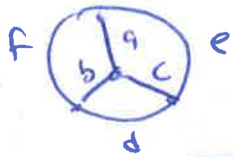
$\neq 0$ ~~iff (a,b,c) r-adm~~.

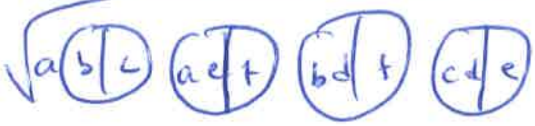


$\neq 0$ iff $(a,b,c), (a,e,f), (b,d,f), (c,d,e)$ r-admi.

Quantum 6j-symbol

$$\begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} =$$



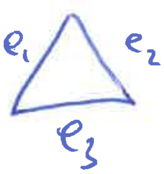


$\in \mathbb{C}.$

~~Rel/Thm~~ (TV, ~~Kauffman-Lins~~).

(M, T) triangulated 3-mfd, closed. V, E, F, T ~~resp.~~ sets of ~~edges, faces,~~ vertices, edges, faces, tetrahedra.

A ~~c~~ coloring $c: E \to I_r$ is r-adm if $\forall f \in F$,

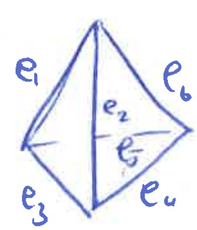
w/ 

, $(c(e_1), c(e_2), c(e_3))$ is r-adm.

For $c \in \mathcal{A}_r$ set of r-adm colorings, let

$$|e|_c = (-1)^{c(e)} [c(e)+1], \quad e \in E$$

$$\|\sigma\|_c = \begin{vmatrix} c(e_1) & c(e_2), & c(e_3) \\ c(e_4) & c(e_5) & c(e_6) \end{vmatrix}, \quad \sigma =$$



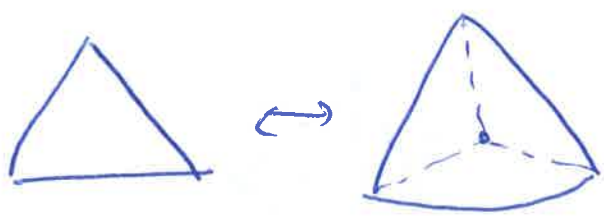
$\in T.$

Def/Thm (TV). For $r \geq 3$, let $\eta = \dfrac{-2r}{(A^2 - A^{-2})^2}$. Then

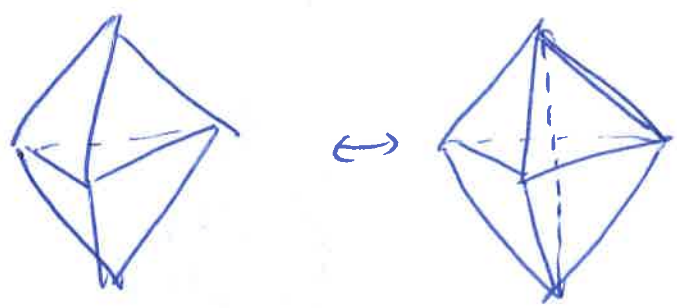
$$TV_r(M) = \eta^{-|V|} \sum_{c \in \mathcal{A}_r} \prod_{e \in E} |e|_c \prod_{\sigma \in T} \|\sigma\|_c$$

defines a real valued invariant of M, i.e., is invariant under 0-2 and 2-3 Pachner moves.

Recall. 0-2



2-3

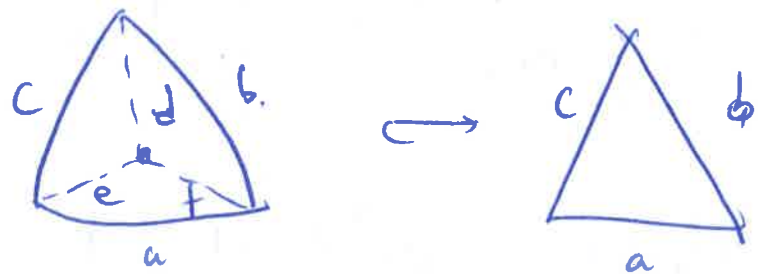


Orthogonality:

$$\sum_s |s||e| \begin{vmatrix} a & b & e \\ c & d & s \end{vmatrix} \begin{vmatrix} a & b & f \\ c & d & s \end{vmatrix} = \delta_{ef}.$$

Cor:

$$\eta^{-1} \sum_{d,e,f} |d||e||f| \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} = 1$$



Biedenharn–Elliot Identity:

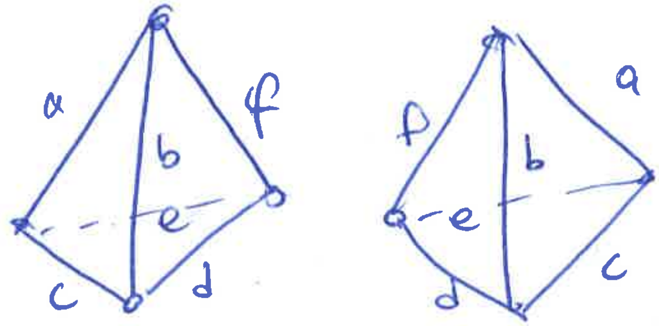
$$\sum_{s} |s| \begin{vmatrix} a & b & i \\ e & d & s \end{vmatrix} \begin{vmatrix} b & c & g \\ f & e & s \end{vmatrix} \begin{vmatrix} c & a & h \\ d & f & s \end{vmatrix} = \begin{vmatrix} a & b & i \\ g & h & c \end{vmatrix} \begin{vmatrix} d & e & i \\ g & h & f \end{vmatrix}.$$

Rm: Can consider M w/ $\partial M \neq \emptyset$ and T ideal triangulation of M.

$$TV_r(M) = \sum_{c \in \mathrm{col}_r} \prod_{e \in E} |e|_c \prod_{\sigma \in T} \|\sigma\|_c .$$

$\Rightarrow$ Invariants of knots / links.

Eg: S^3.



$$TV_r(S^3) = \eta^{-4} \sum_{a,b,\dots,f} |a||b|\cdots|f| \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix}.$$

$$\stackrel{\text{Cor}}{=} \eta^{-3} \sum_{a,b,c} |a||b||c|.$$

$$= \eta^{-3} \sum_{a=0}^{r-2} |a|^2 \left(|a|^{-1} \sum_{b,c} |b||c| \right)$$

$$\stackrel{\text{Lemma},\ a=0}{=} \eta^{-2} \sum_{a=0}^{r-2} |a|^2 = \eta^{-1}. = \boxed{\frac{(A^2 - A^{-2})^2}{-2r}}.$$

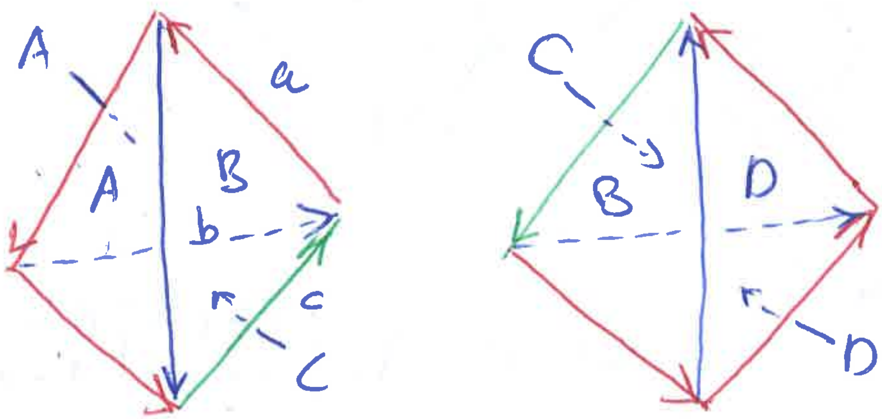
Lemma: $\forall\, a, b \in I_r$,

$$\sum_{c,d} |a|^{-1} |c||d| = \sum_{e,f} |b|^{-1} |e||f|, \quad \text{where}$$

(a,c,d), (b,e,f) r-adm.

Pf: By direct calculation true for $b=a+1$. Then induction

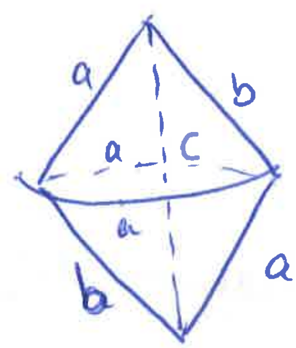
Eg. $S^2 \times S^1$



$$TV_r(S^2\times S^1) = \eta^{-1} \sum_{a,b,c} |a||b||c| \begin{vmatrix} a & a & b \\ a & c & b \end{vmatrix} \begin{vmatrix} a & a & b \\ a & c & b \end{vmatrix},$$

where (a,a,b), (a,b,c) r-adm

$$= \eta^{-1} \sum_{a,b} |b| \left(\sum_c |a||c| \begin{vmatrix} a & a & b \\ a & c & b \end{vmatrix} \begin{vmatrix} a & c & b \\ a & c & b \end{vmatrix} \right)$$



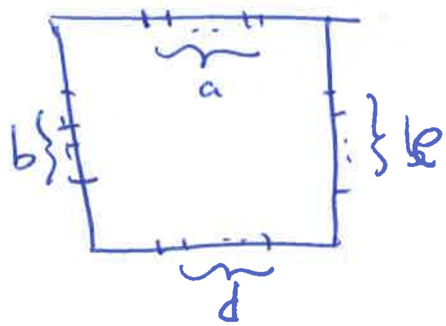
$\overset{\text{Orthogonality}}{\Longrightarrow}$ $\eta^{-1} \sum_{a,b} |b|$ where (a,b) s.t. (a,a,b) r-adm.

direct calculation
$= 1$

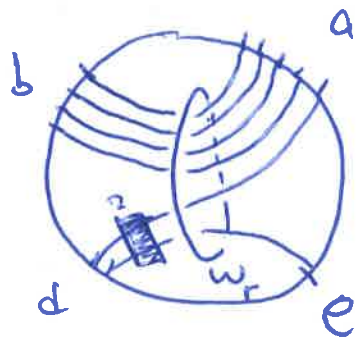
~~Rmk. Can consider M w/ $\partial M \neq \emptyset$, and I ideal triangulations.~~

~~$TV_r(M) = \sum \prod |e| \prod \|\sigma\|$ ... invariant of knots, links.~~

Consider diagrams in $D_{a,b,d,e}$.



$\sim$



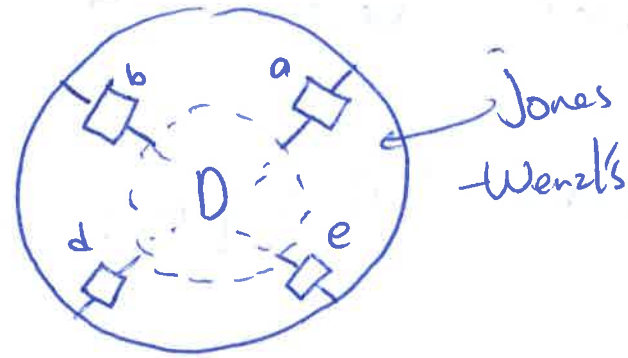
with $a, b, d, e \in I_r$ and $a+b+c+d$ is even.

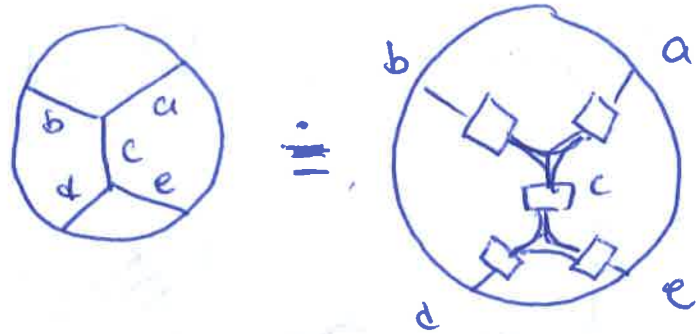
Let $K(D_{a,b,d,e})$ be the $\mathbb{Z}[A_r^{\pm 1}]$-module generated by diagrams in $D_{a,b,d,e}$ modulo ① ②.

① $\times = A \,)(+ A^{-1} \asymp$

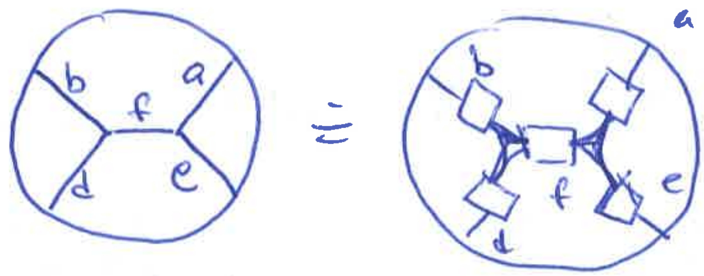
② $O \sqcup D = (-A^2 - A^{-2}) \, D$.

Let $T_{a,b,d,e}$ be the sub-module generated by diagrams of the form



Eg: (i) $Y_c =$  ,

where $c \in I_r$ s.t. (a,b,c), (c,d,e) r-adm.

(ii) $Y'_f =$  ,

where $f \in I_r$ s.t. (a,e,f), (b,d,f) r-adm.

Prop: $\{Y_c\}$ (resp. $\{Y'_f\}$) form a basis of T_{abde}.

Pf: Later.

Cor: Let $B_c = \sqrt{\dfrac{O_c}{(a|b|c)\,(c|d|e)}}\; Y_c$, $B'_f = \sqrt{\dfrac{O_f}{(b|d|f)\,(a|e|f)}}\; Y'_f$.

Then $\{B_c\}$ (resp. $\{B'_f\}$) form a basis of T_{abde}.

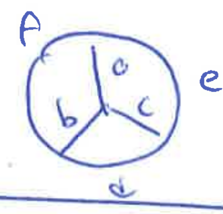
- Let $\left\{\begin{matrix} a & b & c \\ d & e & f \end{matrix}\right\} \in \mathbb{Z}[A^{\pm 1}] \subset \mathbb{C}$ be the unique complex number ~~s.t.~~ (called <u>normalized 6j-symbol</u>) s.t.

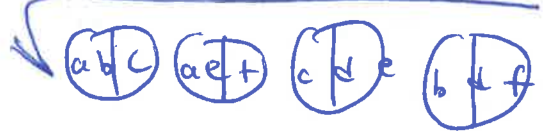
$$B_c = \sum_f \left\{\begin{matrix} a & b & c \\ d & e & f \end{matrix}\right\} B'_f ,$$

where $f \in I_r$ s.t. (b,d,f), (a,e,f) r-adm.

Prop 2:

(i) $\left\{\begin{matrix} a & b & c \\ d & e & f \end{matrix}\right\} = \dfrac{\sqrt{O_c O_f}\ \text{[diagram]}}{\sqrt{\text{[diagrams]}}}$



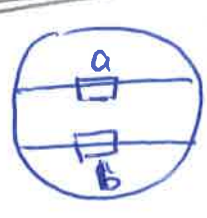


(ii) $\left|\begin{matrix} a & b & c \\ d & e & f \end{matrix}\right| = \dfrac{1}{\sqrt{O_c O_f}} \left\{\begin{matrix} a & b & c \\ d & e & f \end{matrix}\right\}.$

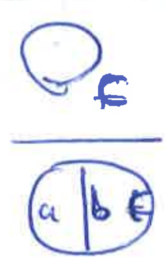
version it later

Pf: Later.

Cor. In T_{aabb},



$= \sum_f$



,

where (a,b,c) r-adm. (Later need it for $TV = |RT|^2$).

Prop 3:

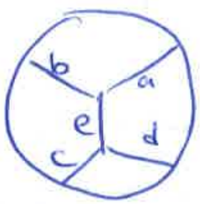
(i) $\displaystyle\sum_s \begin{Bmatrix} a & b & e \\ c & d & s \end{Bmatrix} \begin{Bmatrix} a & b & f \\ c & d & s \end{Bmatrix} = \delta_{ef}$

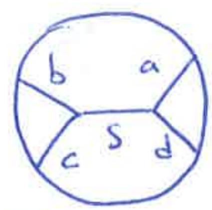
(ii) $\displaystyle\sum_s \begin{Bmatrix} c & b & g \\ e & f & s \end{Bmatrix} \begin{Bmatrix} h & a & c \\ s & f & d \end{Bmatrix} \begin{Bmatrix} a & b & i \\ e & d & s \end{Bmatrix} = \begin{Bmatrix} h & a & c \\ b & g & i \end{Bmatrix} \begin{Bmatrix} h & i & g \\ e & f & d \end{Bmatrix}$.

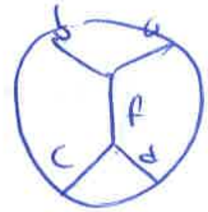
Cor: Orthogonality & BE identity for $\begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix}$.

Pf of Prop 3 assuming Prop [illegible]:

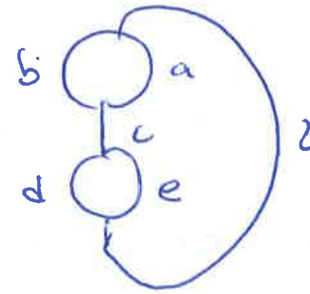
(i)

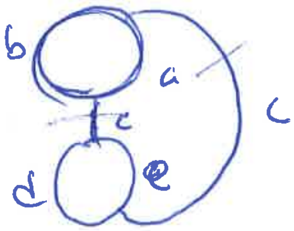
$$\underbrace{\sqrt{\frac{\bigcirc_e}{\theta(a,b,e)\,\theta(c,d,e)}}}$$ 

$$= \sum_s \begin{Bmatrix} a & b & e \\ c & d & s \end{Bmatrix} \underbrace{\sqrt{\frac{\bigcirc_s}{\theta(a,d,s)\,\theta(b,c,s)}}}$$ 

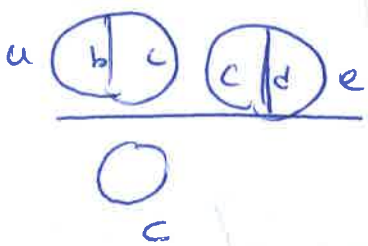
$$= \sum_s \sum_f \underline{\begin{Bmatrix} a & b & e \\ c & d & s \end{Bmatrix} \begin{Bmatrix} a & b & f \\ c & d & s \end{Bmatrix}} \sqrt{\frac{\bigcirc_f}{\theta(a,b,f)\,\theta(c,d,f)}}$$ 

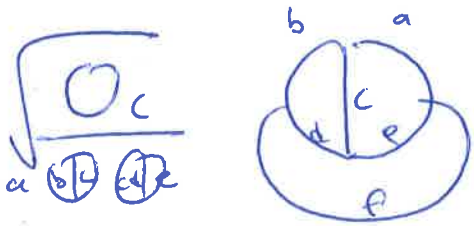
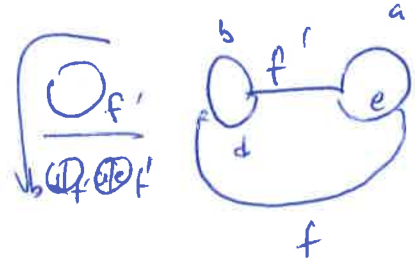
$$\Rightarrow \sum_s \begin{Bmatrix} a & b & e \\ c & d & s \end{Bmatrix} \begin{Bmatrix} a & b & f \\ c & d & s \end{Bmatrix} = \delta_{ef}.$$

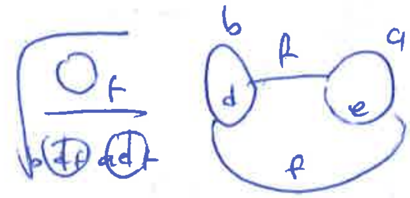
$\sum_{c} k_c \cdot$  $= 0.$

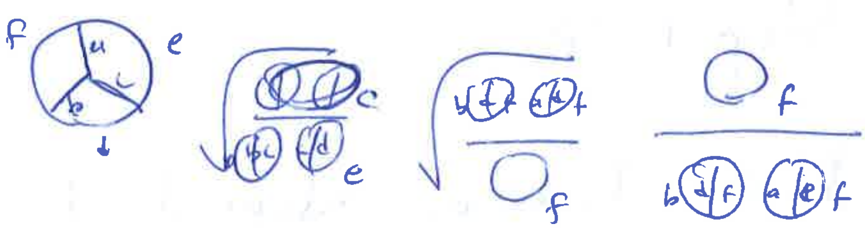
By Lemma, k_c  $= 0.$

//

and k_c  $\Rightarrow k_c = 0.$

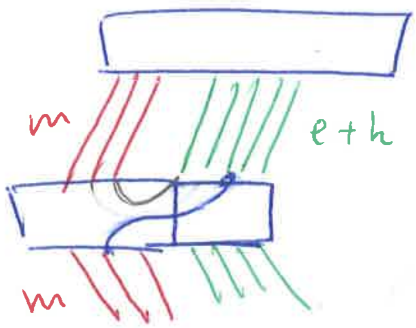
Prop 2 (ii):  $= \sum_{f'} \begin{Bmatrix} a & b & c \\ d & e & f' \end{Bmatrix}$ 

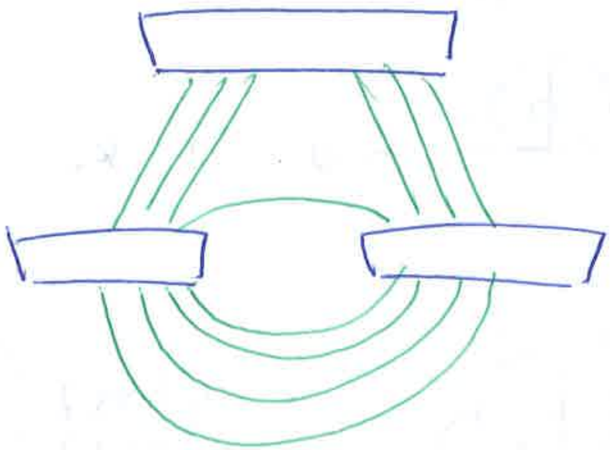
$\overset{\text{Lemma}}{=} \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$ 

$\Rightarrow \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} =$  $=$ RHS.

~~Claim: for each~~

(13)

If  , some green is connected w/ red, then I turn around, since there are more "in" than "out".

Therefore,  , which contains a "turn-around", since there is no "out",

$\Rightarrow$ $\cdot = 0$, since  kills turn-arounds $\square$.

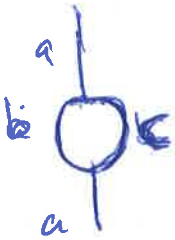
Pf of Prop 1. 2.

- $\{Y_c\}$ are linearly independent.

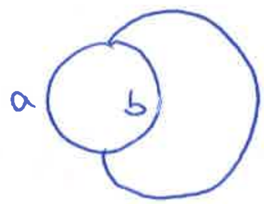
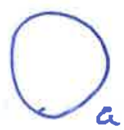
Suppose $\sum_c k_c Y_c = 0$, then $\forall i \in I_{r-2}$ s.t.

(a,b,c), (d,e,i) r-adm,

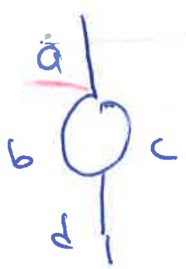
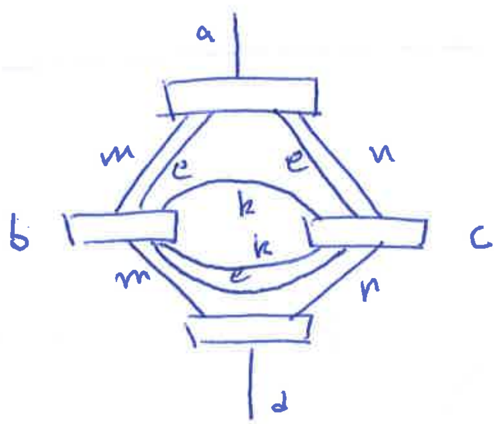
Pf. If $a = d$, then  $\in TL_a$, and

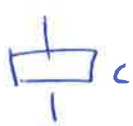
 $= k \cdot$  for some $k \in \mathbb{C}$.

Closing it up,

 $= k \cdot$  $\Rightarrow k =$  / .

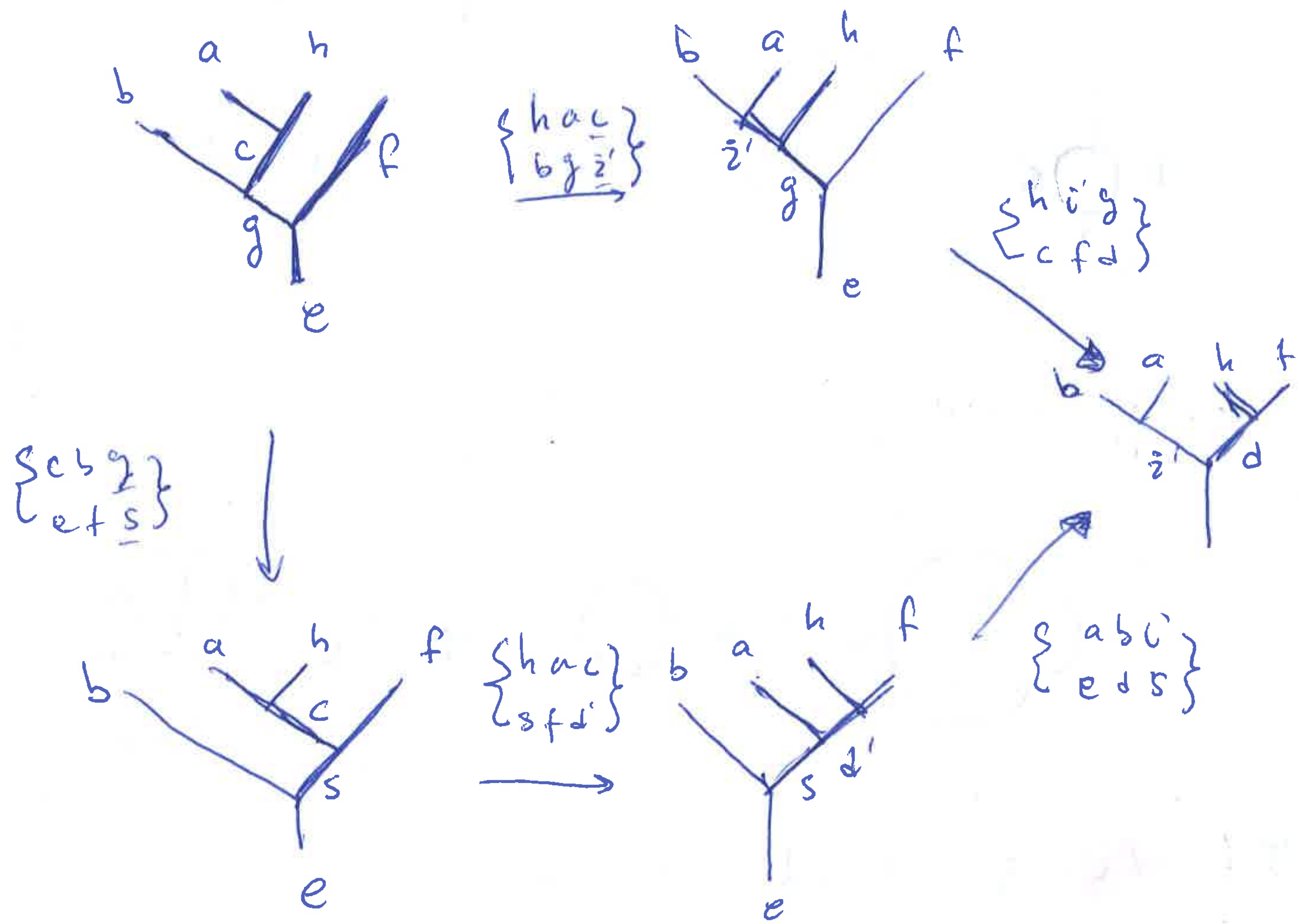
If w.l.o.g. $a > d$, then

 = , where $e = \frac{a-d}{2}$.

Each  or  is a linear combination of embedded diagrams 

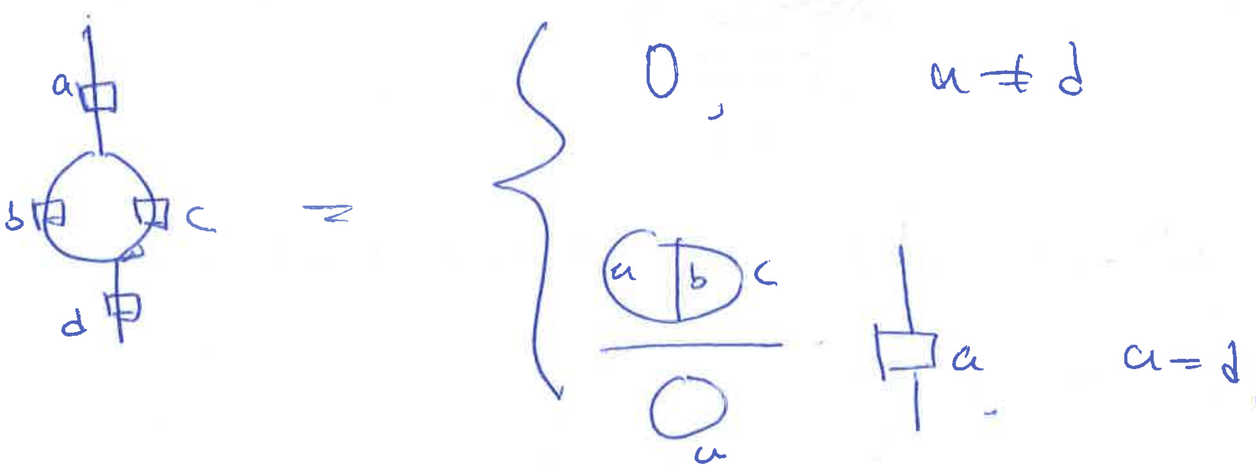
Claim: for each ↗, there is a "turn-around".

(ii) Similar, consider

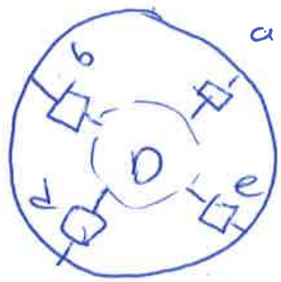


~~Cor of Orthogonality:~~ 16

Lemma:

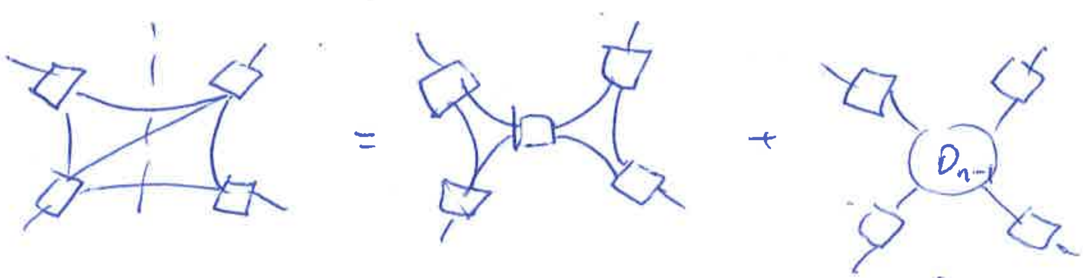


• $\{\Psi_c\}$ span T_{abcd}.

Each element of T_{abcd} is a linear combination of diagrams as  w/ D embedded, and is non-zero iff  or 

since otherwise, $\exists$ turn-arounds.

Since  $+ D_{n-1}$, where D_{n-1} has "turn arounds":



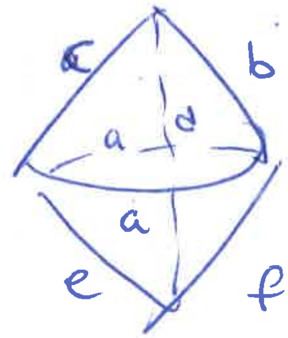
, where

D_{n-1} has less intersection w/ than n.

Induction $\Rightarrow$ result. $\square$

Pf of Lem:

By Orthogonality $(a,b,c)\ (a,e,f)$ r-adm.



$$\sum_d |d||a| \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} = 1.$$

$(b,d,f)\ (c,d,e)$ r-adm.

$$\Rightarrow \eta^{-1} \sum_{d,e,f} |d||e||f| \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix}$$

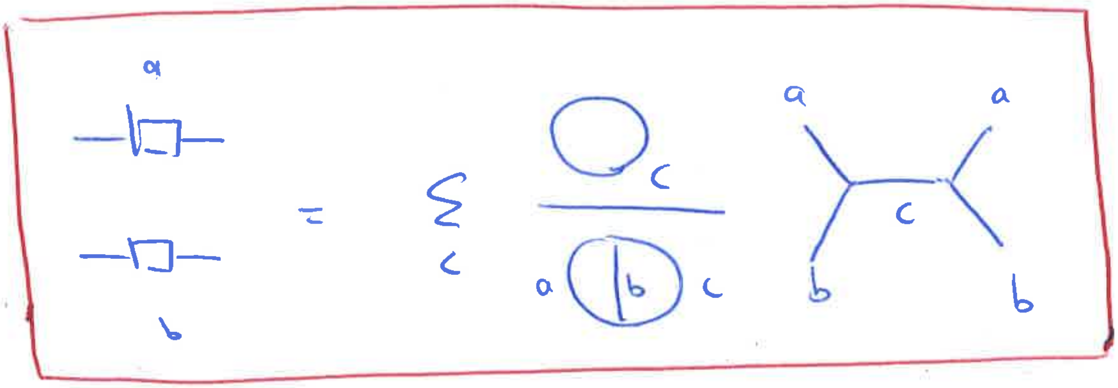
$$= \eta^{-1} \sum_{e,f} |e||f||a|^{-1} \left(\sum_d |d||a| \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} \right).$$

$$= \eta^{-1} \boxed{\sum_{e,f} |e||f||a|^{-1}}$$

Lemma.

$$= \eta^{-1} \cdot \eta = 1.$$

Fusion Rule:

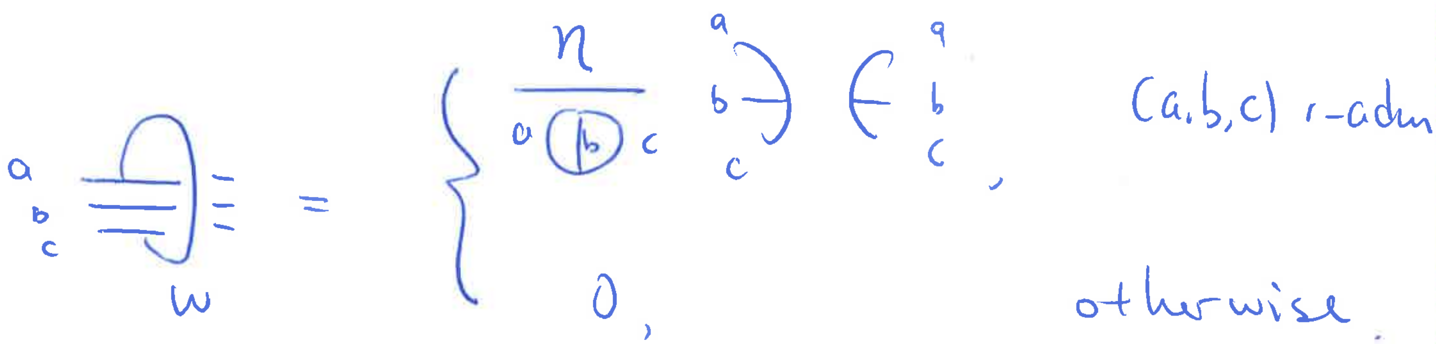


where $c \in I_r$ s.t (a,b,c) is r-adm.

pf. Consider T_{aabb}, then

□.

Cor. $\eta = \langle \omega_r \rangle$. Then



pf. By Fushion Rule.