

last time.

For $r \geq 3$, let $A_r = e^{\frac{\pi i}{2r}}$ and $I_r = \{0, 1, \dots, \frac{r-2}{2}\}$

$(a, b, c) \in I_r^3$ is ~~an~~ r -admissible if

(i) $a+b \geq c, b+c \geq a, c+a \geq b.$

(ii) $a+b+c \leq 2(r-2),$

(iii) $a+b+c$ is even.

$$a \circlearrowleft = \langle a \circlearrowleft \rangle = (-1)^a [n+1]$$

$$a \begin{array}{|c|} \hline b \\ \hline c \\ \hline \end{array} = \langle \begin{array}{c} \text{diagram with } a, b, c, k, m, n \end{array} \rangle \neq 0 \iff \text{if } (a, b, c) \text{ is } r\text{-admissible.}$$

$l = \frac{a+b-c}{2}, m = \frac{b+c-a}{2}, n = \frac{a+c-b}{2}$

$$A \begin{array}{c} \circlearrowleft \\ \hline \begin{array}{|c|} \hline a \\ \hline b \\ \hline c \\ \hline \end{array} \\ \hline \circlearrowleft \end{array} \neq 0 \iff (a, b, c), (a, e, f), (b, d, f), (c, d, e)$$

r -admi.


Quantum 6j-symbol

$$\begin{array}{|c|} \hline abc \\ \hline det \\ \hline \end{array} = \frac{\text{Diagram of a tetrahedron with vertices a, b, c, d and faces f, e}}{\sqrt{\text{Diagram of four faces: } \begin{array}{|c|} \hline abc \\ \hline \end{array} \begin{array}{|c|} \hline det \\ \hline \end{array} \begin{array}{|c|} \hline bdt \\ \hline \end{array} \begin{array}{|c|} \hline cde \\ \hline \end{array}}} \in \mathbb{C}$$

~~Definition (TV, Kauffman-Lins)~~

(M, T) triangulated 3-mtd, closed. V, E, T ~~resp.~~^F
 sets of ~~edges~~, ~~faces~~ vertices, edges, tetrahedra.
_{faces}

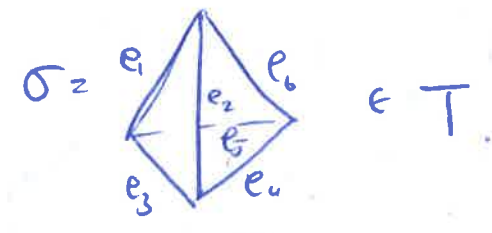
A coloring $c: E \rightarrow \mathbb{I}_r$ is r -adm if $\forall f \in F$,

w/  $(c(e_1), c(e_2), c(e_3))$ is r -adm.

For $c \in \mathcal{A}_r$ set of r -adm colorings, let

$$|e|_c = (-1)^{c(e)} [c(e) + 1], \quad e \in E$$

$$\| \sigma \|_c = \begin{vmatrix} c(e_1) & c(e_2) & c(e_3) \\ c(e_4) & c(e_5) & c(e_6) \end{vmatrix},$$



Def / Thm (TV). For $r \geq 3$, let $\eta = \frac{-2r}{(A^2 - A^{-2})^2}$. Then

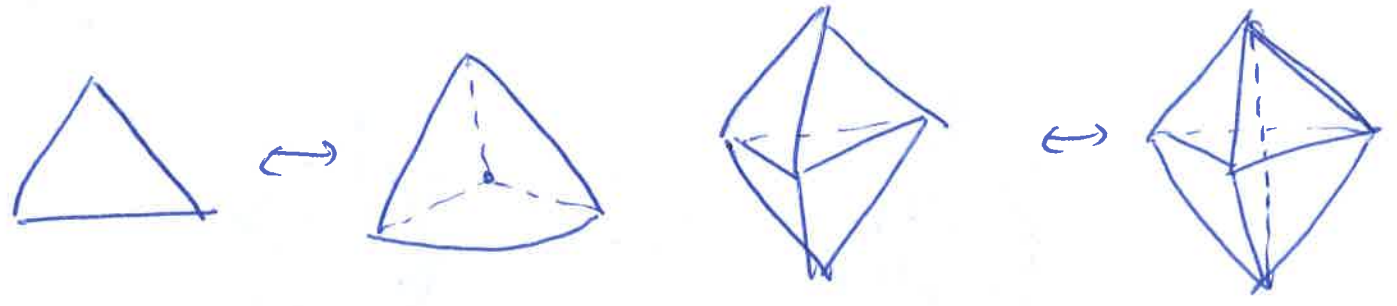
$$TV_r(M) = \eta^{-|V|} \sum_{c \in \mathcal{A}_r} \prod_{e \in E} |e|_c \prod_{\sigma \in T} \|\sigma\|_c$$

defines a real valued invariant of M , i.e., is invariant under 0-2 and 2-3 Pachner Moves.

Recall:

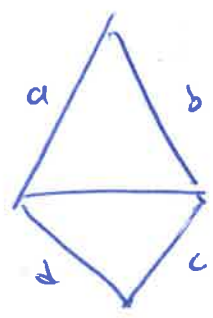
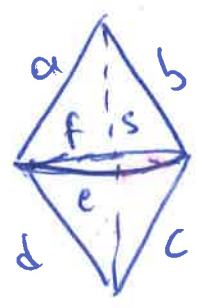
0-2

2-3



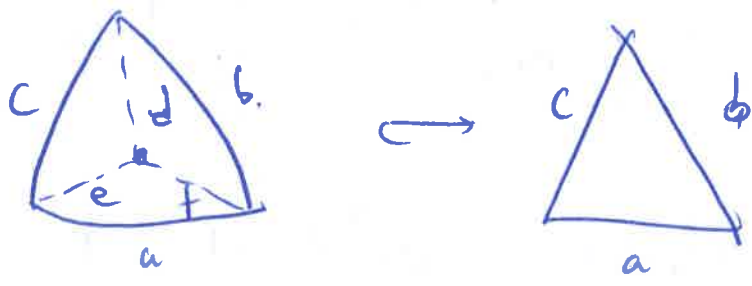
Orthogonality:

$$\sum_s |s| |e| \begin{vmatrix} a & b & e \\ c & d & s \end{vmatrix} \begin{vmatrix} a & b & f \\ c & d & s \end{vmatrix} = \det.$$



Cor:

$$\sum_{d \in \mathcal{E}} |d| |e| |f| \frac{|abc|}{|def|} \frac{|abc|}{|def|} = 1$$



Bredenkarn-Elliot Identity:

$$\sum_s |s| \begin{vmatrix} abc \\ eds \end{vmatrix} \begin{vmatrix} bfg \\ fes \end{vmatrix} \begin{vmatrix} cah \\ dfs \end{vmatrix} = \begin{vmatrix} abc \\ ghc \end{vmatrix} \begin{vmatrix} dec \\ ghf \end{vmatrix}$$

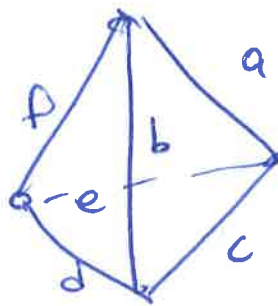
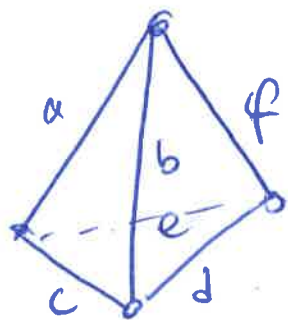


km: Can consider $m \cup \partial m \neq \emptyset$ and \mathcal{T} ideal triangulation of m .

$$TV_r(m) = \sum_{c \in \mathcal{C}_r} \prod_{e \in E} |e|_c \prod_{\sigma \in \mathcal{T}} ||\sigma||_c$$

\Rightarrow Invariants of knots / links.

Eg: S^3 .



(5)

$$TV_r(S^3) = \eta^{-4} \sum_{a, b, \dots, f} |a||b| \dots |f| \begin{vmatrix} a & b & c \\ \dots & \dots & \dots \end{vmatrix} \begin{vmatrix} a & b & c \\ \dots & \dots & \dots \end{vmatrix}$$

$$\stackrel{\text{Cor}}{=} \eta^{-3} \sum_{a, b, c} |a||b||c|$$

$$= \eta^{-3} \sum_{a=0}^{r-2} |a|^2 \left(|a|^r \sum_{b, c} |b||c| \right)$$

Lemma: $\stackrel{a=0}{=} \eta^{-2} \sum_{a=0}^{r-2} |a|^2 = \eta^{-1} = \boxed{\frac{(A^r - A^{-r})^2}{-2r}}$

Lemma: $\forall a, b \in I_r,$

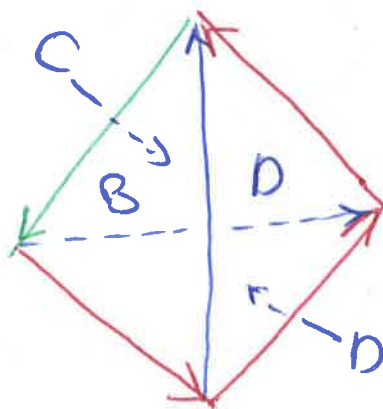
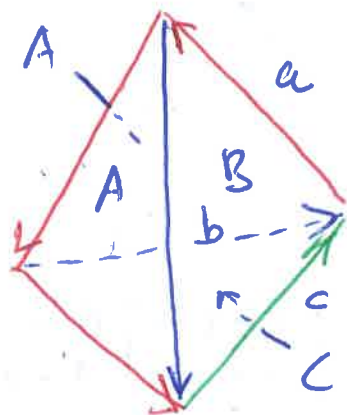
$$\sum_{c, d} |a|^r |c||d| = \sum_{e, f} |b|^r |e||f|, \quad \text{where}$$

$(a, c, d), (b, e, f)$ r -adm.

Pf: By direct calculation true for $b=a+1$, Then induction

Ex. $S^2 \times S^1$

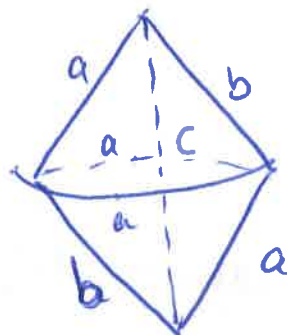
(6)



$$TV_r(S^2 \times S^1) = \eta^{-1} \sum_{a,b,c} \left\{ |a| |b| |c| \begin{vmatrix} a & a & b \\ a & c & b \end{vmatrix} \begin{vmatrix} a & a & b \\ a & c & b \end{vmatrix} \right\}$$

where $(a, a, b), (a, b, c)$ r-adm

$$= \eta^{-1} \sum_{a,b} |b| \left(\sum_c |a| |c| \begin{vmatrix} a & a & b \\ a & c & b \end{vmatrix} \begin{vmatrix} a & c & b \\ a & c & b \end{vmatrix} \right)$$



Orthogonality

$$\eta^{-1} \sum_{a,b} |b|$$

where (a, b) s.t

(a, a, b) r-adm

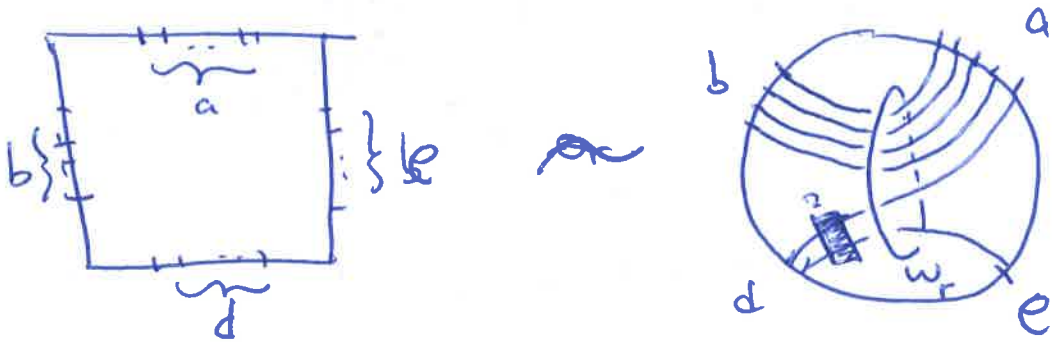
direct calculation
= 1

~~RM: Consider M w/ $\partial M \neq \emptyset$, and \mathcal{I} ideal triangulations.~~

~~$$TV_r(M) = \sum_{c \in \mathcal{I}} \prod_{e \in E} |e| \prod_{o \in T} \#o|c$$~~

~~FW: Invariant of knots, links.~~

Consider diagrams in $D_{a,b,d,e}$.



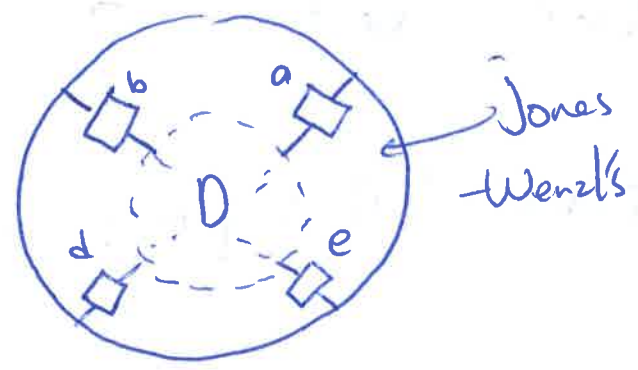
with $a, b, d, e \in \mathbb{I}_r$ and $a+b+c+d$ is even.

Let $k(D_{a,b,d,e})$ be the $\mathbb{Z}[A_r^{\pm 1}]$ -module generated by diagrams in $D_{a,b,d,e}$ modulo ① ②.

① $\diagdown = A \diagup + A^{-1} \cup$

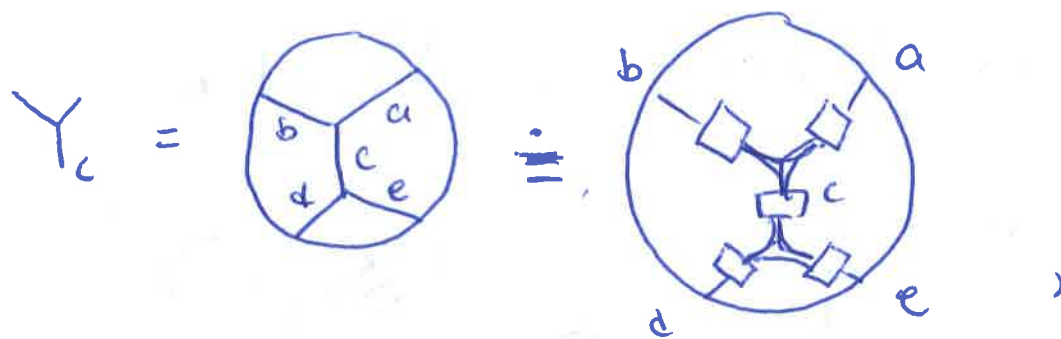
② $\bigcirc \cup D = (A^2 - A^{-2}) D$

Let $T_{a,b,d,e}$ be the sub-module generated by diagrams of the form



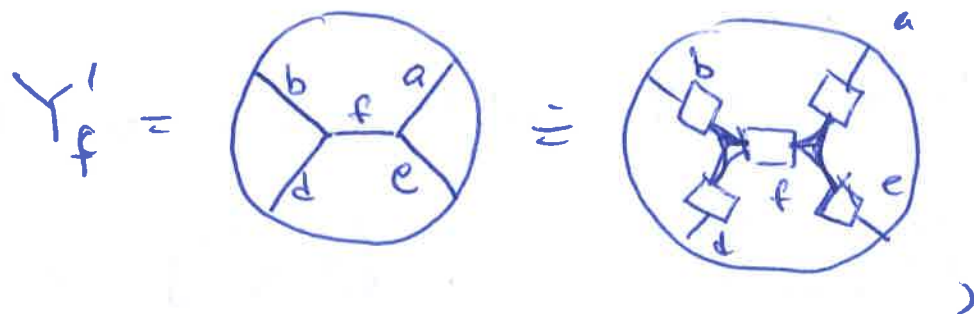
Jones-Wenzl's

Eg: (i)



where $c \in I_r$ s.t. $(a, b, c), (c, d, e)$ r -adm.

(ii)



where $f \in I_r$ s.t. $(a, e, f), (b, d, f)$ r -adm.

~~Prop 1~~ Prop 1: $\{Y_c\}$ (resp. $\{Y'_f\}$) form a basis of T_{abde} .

pf: later.

Cor: let $B_c = \frac{O_c}{\left(\begin{smallmatrix} a|bc & c|d \\ e \end{smallmatrix} \right)} Y_c$, $B'_f = \frac{O_f}{\left(\begin{smallmatrix} b|d & d|e \\ a|c \end{smallmatrix} \right)} Y'_f$.

Then $\{B_c\}$ (resp. $\{B'_f\}$) form a basis of T_{abde} .

Let $\left\{ \begin{matrix} abc \\ det \end{matrix} \right\} \in \mathbb{Z}[A^{\pm 1}] \subset \mathbb{C}$ be the unique complex number ~~set~~ (called normalized bj-symbol) sit.

$$B_c = \sum_f \left\{ \begin{matrix} abc \\ det \end{matrix} \right\} B'_f,$$

where $f \in I_r$ sit $(b, d, f), (a, e, f)$ r-adm.

Prop 2:

(i) $\left\{ \begin{matrix} abc \\ det \end{matrix} \right\} = \frac{\sqrt{O_c O_f} \begin{matrix} A \\ \circlearrowleft \\ \begin{matrix} a & b & c \\ & d & e \end{matrix} \\ e \end{matrix}}{\begin{matrix} \circlearrowleft \\ \begin{matrix} a|c & a|e & c|d & e & b|d & f \end{matrix} \end{matrix}}$

(ii) $\left| \begin{matrix} abc \\ det \end{matrix} \right| = \frac{1}{\sqrt{O_c O_f}} \left\{ \begin{matrix} abc \\ det \end{matrix} \right\}$

Mention it later

pf: later.

Cor: In $Taabb$,

$$\begin{matrix} \circlearrowleft \\ \begin{matrix} a \\ \square \\ b \end{matrix} \end{matrix} = \sum_f \frac{\begin{matrix} \circlearrowleft \\ f \end{matrix}}{\begin{matrix} \circlearrowleft \\ a|b|e \end{matrix}} \begin{matrix} \circlearrowleft \\ \begin{matrix} a & e & b \\ & a & b \end{matrix} \end{matrix}$$

where (a, b, c) r-adm. (later need it for $trv = (2\pi)^2$).

Prop 3:

$$(i) \sum_s \begin{Bmatrix} a b e \\ c d s \end{Bmatrix} \begin{Bmatrix} a b f \\ c d s \end{Bmatrix} = \delta_{ef}$$

$$(ii) \sum_s \begin{Bmatrix} c b g \\ e t s \end{Bmatrix} \begin{Bmatrix} h a c \\ s t d \end{Bmatrix} \begin{Bmatrix} a b u \\ e d s \end{Bmatrix} = \begin{Bmatrix} h a c \\ b g u \end{Bmatrix} \begin{Bmatrix} h i g \\ e t d \end{Bmatrix}$$

Cor: Orthogonality & BE identity for $|abc\rangle$

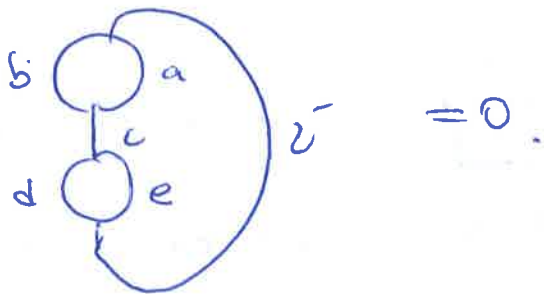
Pf of Prop 3 (assuming Prop 1):

$$(i) \sum_s \begin{Bmatrix} a b e \\ c d s \end{Bmatrix} \begin{Bmatrix} a b f \\ c d s \end{Bmatrix} = \sum_s \begin{Bmatrix} a b e \\ c d s \end{Bmatrix} \begin{Bmatrix} a b f \\ c d s \end{Bmatrix} \begin{Bmatrix} a b e \\ c d s \end{Bmatrix} \begin{Bmatrix} a b f \\ c d s \end{Bmatrix}$$

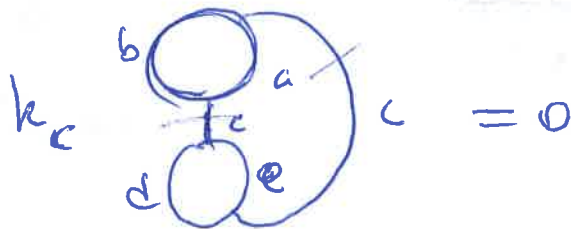
$$= \sum_s \sum_t \begin{Bmatrix} a b e \\ c d s \end{Bmatrix} \begin{Bmatrix} a b f \\ c d s \end{Bmatrix} \begin{Bmatrix} a b e \\ c d t \end{Bmatrix} \begin{Bmatrix} a b f \\ c d t \end{Bmatrix}$$

$$\Rightarrow \sum_s \begin{Bmatrix} a b e \\ c d s \end{Bmatrix} \begin{Bmatrix} a b f \\ c d s \end{Bmatrix} = \delta_{ef}$$

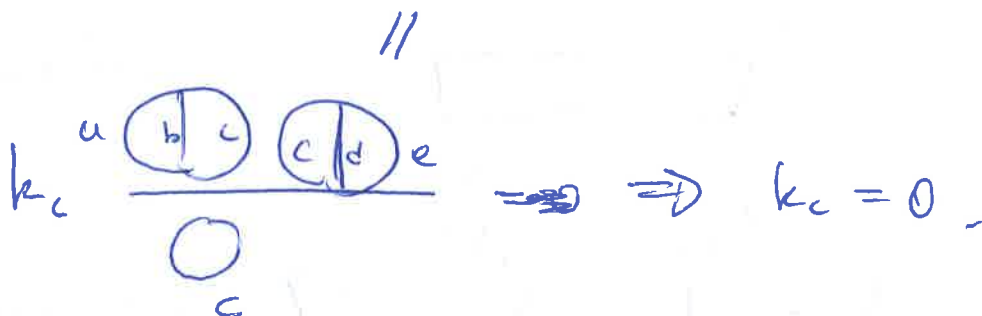
$$\sum_c k_c$$



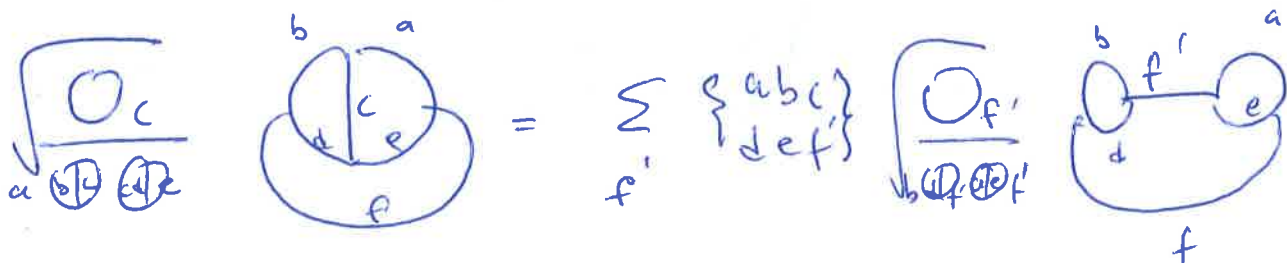
By lemma 1)



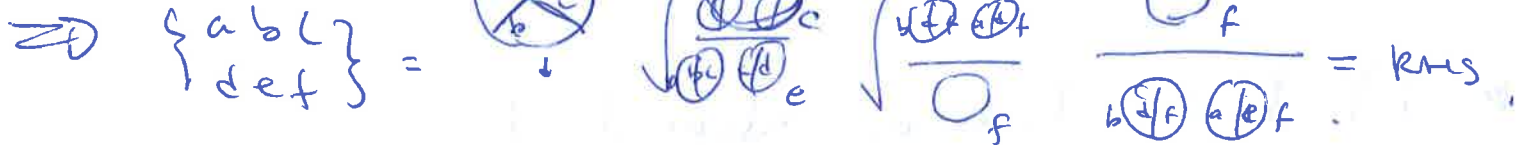
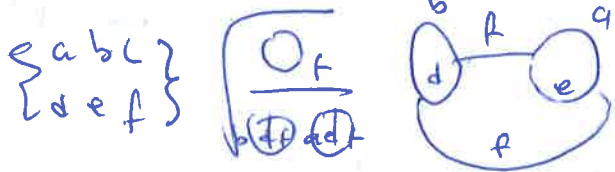
and



Prop 2 (i):



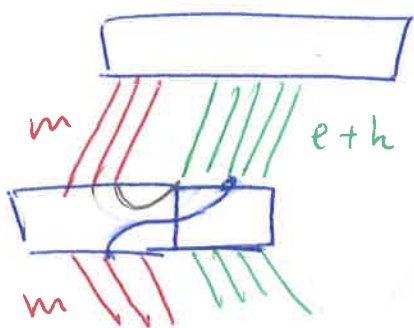
lemma



Claim: for each

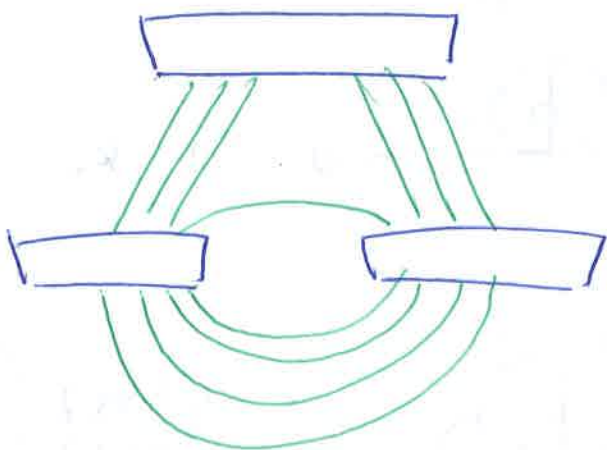
(13)

If



, some green is connected w/ red, then \exists turn around, since there are more "in" than "out".

Therefore,



, which contains a "turn-around". Since there is no "out".

$\Rightarrow \cdot = 0$, since \square kills turn-arounds \square .

Pf of Prop 1, 2.

$\bullet \{Y_c\}$ are linearly independent.

Suppose $\sum_{c=1}^n k_c Y_c = 0$, then $\forall i \in I_{r-1}$ sit

(a,b,c), (d,e,i) r-adm,

pf. If $a=d$, then $b \begin{array}{c} a \\ \circ \\ a \end{array} c \in TLa$, and

$$b \begin{array}{c} a \\ \circ \\ a \end{array} c = k \cdot \begin{array}{c} | \\ \square \\ | \end{array} \text{ for some } k \in \mathbb{C}.$$

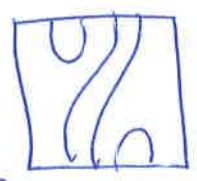
Closing it up,

$$a \begin{array}{c} \circ \\ \circ \\ \circ \end{array} c = k \cdot \begin{array}{c} \circ \\ \circ \end{array} a \Rightarrow k = \frac{a \begin{array}{c} | \\ \circ \\ | \end{array} c}{O_a}$$

If w.l.o.g. $a > d$, then

$$b \begin{array}{c} \bar{a} \\ \circ \\ d \end{array} c = \begin{array}{c} a \\ | \\ \begin{array}{c} m \quad e \quad e \quad n \\ \circ \\ k \quad k \\ \circ \\ e \quad e \quad m \quad n \\ | \\ d \end{array} \end{array} c, \text{ where } e = \frac{a-d}{2}$$

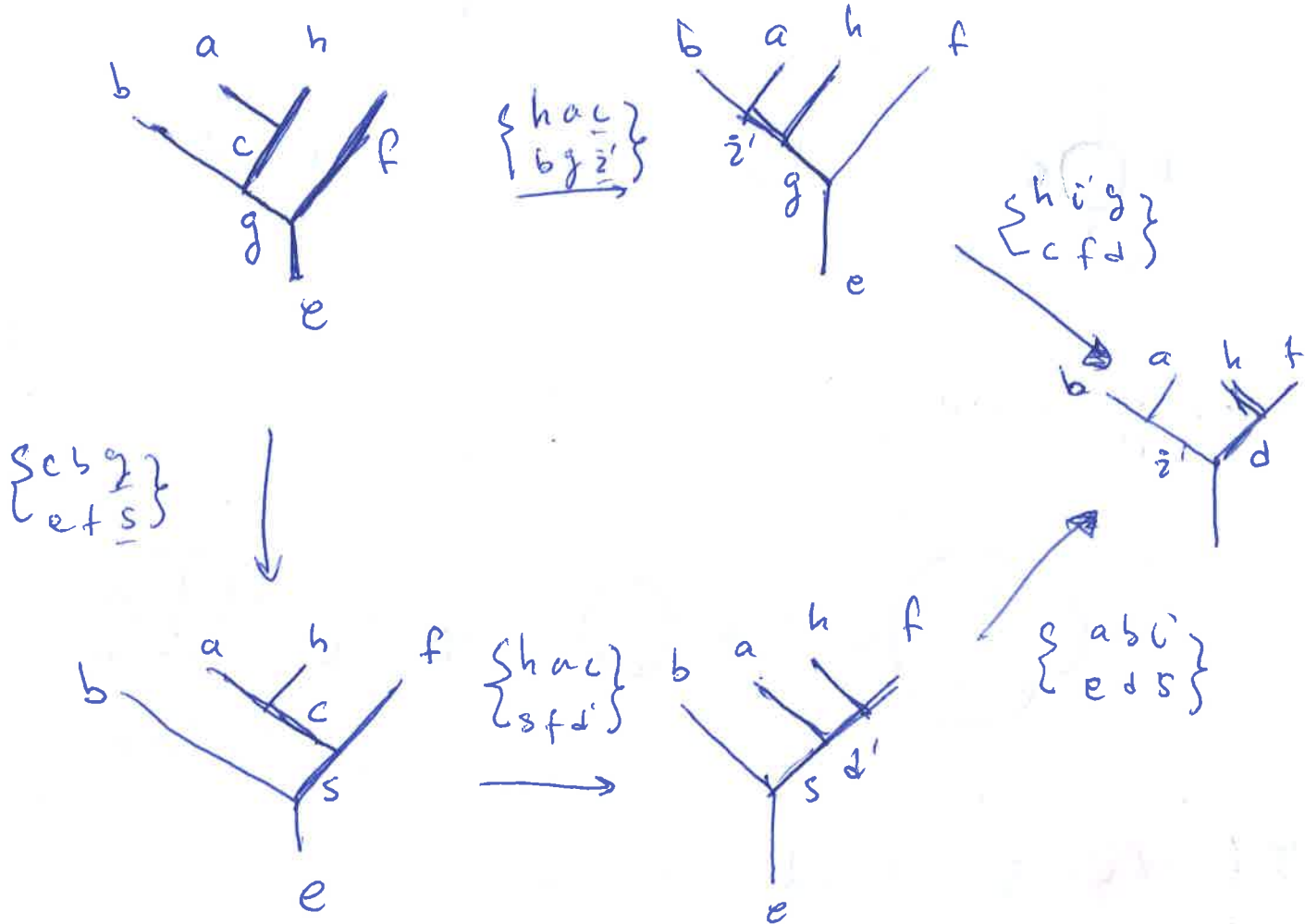
Each $\begin{array}{c} | \\ \square \\ | \end{array} b$ or $\begin{array}{c} | \\ \square \\ | \end{array} c$ is a linear combination of embedded digrams



Claim: for each $\begin{array}{c} | \\ \square \\ | \end{array}$, there is a "turn-around".

(ii) Similar, consider

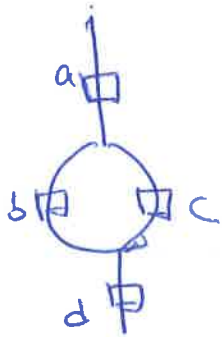
(11)



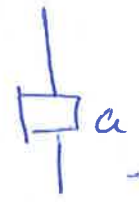
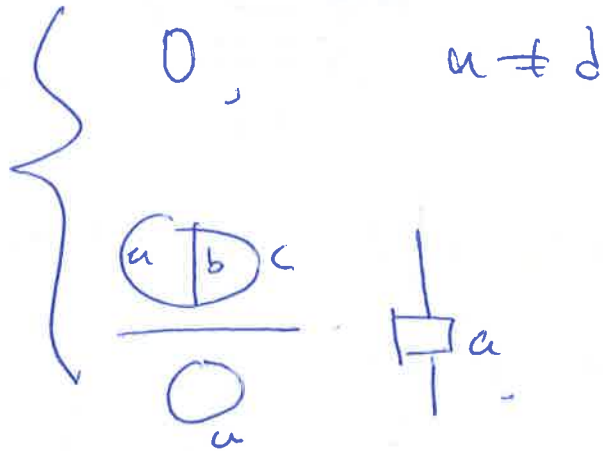
Cor of Diagonalizability:

~~(16)~~ (16)

Lemma:



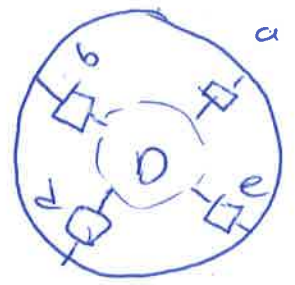
\approx



$a = d$

$\{T_c\}$ span T_{abcd} .

Each element of T_{abcd} is a linear combination of diagrams of



w/ D embedded, and

is non-zero iff



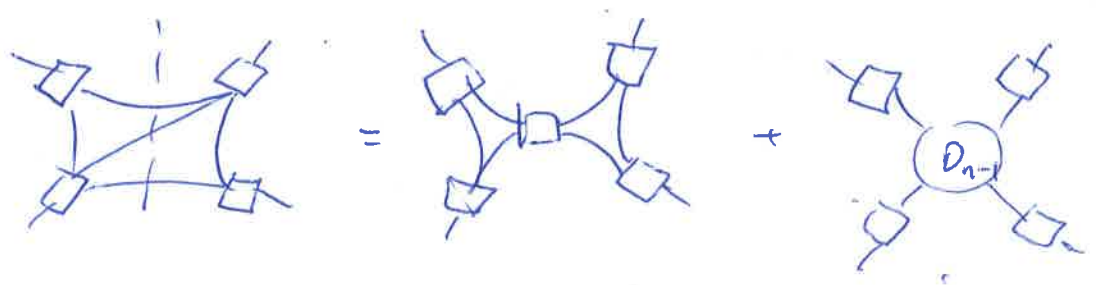
or



Since otherwise, \exists turn-arounds.

Since $\frac{1_n}{\vdots} = \square_n^f + D_{n-1}$, where

D_{n-1} has "turn arounds".

 , where

D_{n-1} has less intersection w/ \vdots than n .

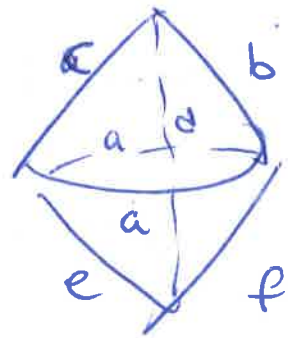
Induction \Rightarrow result.

Pf of Lor:

By Orthogonality $(a, b, c) (a, e, f)$ r -adm.

(16)

$$\sum_d |d| |a| \left| \frac{abc}{det} \right| \left| \frac{abc}{det} \right| = 1.$$



$(b, d, f) (c, d, e)$ r -adm.

$$\Rightarrow \eta^{-1} \sum_{d, e, f} |d| |e| |f| \left| \frac{abc}{det} \right| \left| \frac{abc}{det} \right|$$

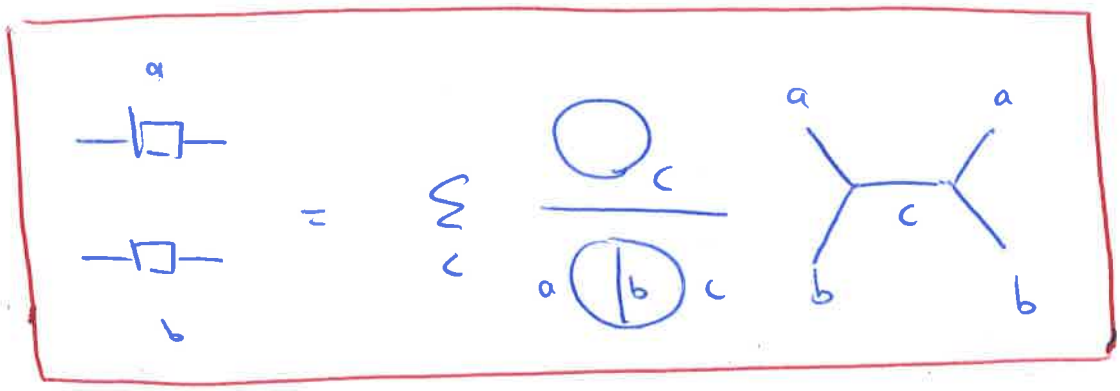
$$= \eta^{-1} \sum_{e, f} |e| |f| |a|^{-1} \left(\sum_d |d| |a| \left| \frac{abc}{det} \right| \left| \frac{abc}{det} \right| \right)$$

$$= \eta^{-1} \sum_{e, f} |e| |f| |a|^{-1}$$

Cancel.

$$\Rightarrow \eta^{-1} \cdot \eta = 1.$$

Fusion Rule:



where $c \in I_r$ s.t. (a, b, c) is r -adm.

pt: Consider T_{aabb} , then $\begin{matrix} a \\ \parallel \\ b \end{matrix} = \begin{matrix} a & a \\ & \diagdown \diagup \\ & 0 \\ & \diagup \diagdown \\ b & b \end{matrix}$ \square

Cor: $\eta = \langle w_r \rangle$. Then

$$\begin{matrix} a \\ \parallel \\ b \\ \parallel \\ c \end{matrix} \Bigg|_w = \begin{cases} \frac{\eta}{a \oplus (b \oplus c)} \begin{matrix} a \\ \parallel \\ b \\ \parallel \\ c \end{matrix} \in \begin{matrix} a \\ b \\ c \end{matrix}, & (a, b, c) \text{ r-adm} \\ 0, & \text{otherwise} \end{cases}$$

pt: By Fusion Rule.

$$\begin{matrix} a \\ \parallel \\ b \\ \parallel \\ c \end{matrix} \Bigg|_w = \sum_d \frac{\text{circle}_d}{a \oplus (b \oplus d)} \begin{matrix} a \\ \parallel \\ b \\ \parallel \\ c \end{matrix} \Bigg|_d = \sum_f \sum_d \frac{\text{circle}_d \text{ circle}_f}{a \oplus (b \oplus d) \oplus (c \oplus f)} \begin{matrix} a \\ \parallel \\ b \\ \parallel \\ c \end{matrix} \Bigg|_w$$

Recall $\begin{matrix} a \\ \parallel \\ b \\ \parallel \\ c \end{matrix} \Bigg|_w = \begin{cases} D^{w_r}, & n=0 \\ 0, & n \neq 0 \end{cases}$

$$\frac{\text{circle}_c \text{ circle}_0}{a \oplus (b \oplus c) \oplus 0} \begin{matrix} a \\ \parallel \\ b \\ \parallel \\ c \end{matrix} \Bigg|_w = \frac{\eta}{a \oplus (b \oplus c)} \begin{matrix} a \\ \parallel \\ b \\ \parallel \\ c \end{matrix} \in \begin{matrix} a \\ b \\ c \end{matrix} \quad (a, b, c) \text{ r-adm}$$

otherwise \square

