The Skein Algebra of Arcs and Links and the Decorated Teichmüller Space

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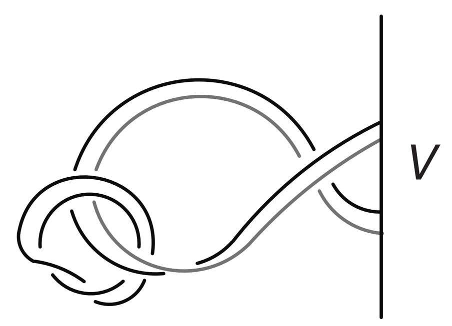
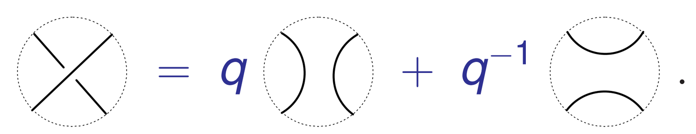
The Skein Algebra $\mathcal{AS}_h(\Sigma)$ of Arcs and Links

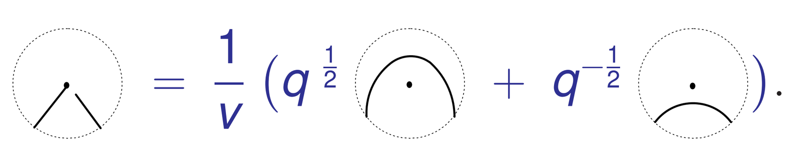
- Let Σ be an oriented punctured surface with V the set of punctures.
- A generalized framed link in $\Sigma \times [0,1]$,



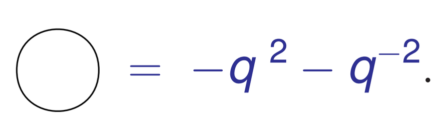
- **Definition** Let $\mathbb{C}[[h]]$ be the ring of power series in h and let $q = e^{\frac{h}{4}} \in \mathbb{C}[[h]]$. The skein algebra of arcs and links $\mathcal{AS}_h(\Sigma)$ is the $\mathbb{C}[[h]]$-module generated by $\mathcal{L}$, the generalized framed links in $\Sigma \times [0,1]$ and $V^{\pm}$, the punctures of Σ and their formal inverses, modulo the following relations:
  - Kauffman bracket skein relation:



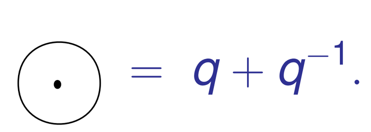
  - Puncture-skein relation:



  - Framing relation:



  - Puncture relation:

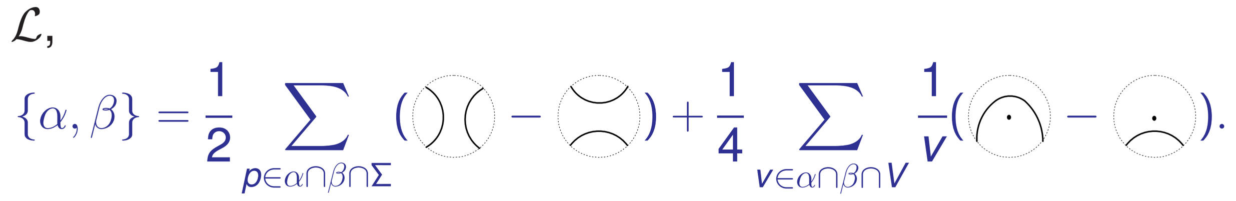


When $V = \emptyset$, $\mathcal{AS}_h(\Sigma)$ coincides with the Kauffman bracket skein algebra $\mathcal{S}_h(\Sigma)$ defined by Przytycki and Turaev.

- Multiplication $\cdot$ on $\mathcal{AS}_h(\Sigma)$:
  - $\forall\, \alpha, \beta \in \mathcal{L}$, $\alpha \cdot \beta =$ stacking α above β along the direction of $[0,1]$;
  - $\forall\, v \in V$ and $\forall\, \alpha \in \mathcal{L}$, $v \cdot \alpha = \alpha \cdot v$ and $v \cdot v^{-1} = v^{-1} \cdot v = 1$.

$\mathcal{AS}_0(\Sigma)$ and the Generalized Goldman Bracket

- When $h = 0$, the multiplication $\cdot$ is commutative. $\mathcal{AS}_0(\Sigma)$ can be considered as generated by the projections of the generalized framed links, i.e., immersed loops and arcs on the Σ.
- Generalized Goldman bracket $\{,\}$ on $\mathcal{AS}_0(\Sigma)$:
  - $\forall\, v \in V$ and $\forall\, \alpha \in \mathcal{L}$,
$$\{v^{\pm 1}, \alpha\} = 0,$$
  - $\forall\, \alpha, \beta \in \mathcal{L}$,



Deformation Quantization

- **Theorem 1**
(1) $(\mathcal{AS}_h(\Sigma), \cdot)$ is a well-defined associative $\mathbb{C}[[h]]$-algebra,
(2) $(\mathcal{AS}_0(\Sigma), \cdot, \{,\})$ is a well-defined Poisson algebra,
(3) $(\mathcal{AS}_h(\Sigma), \cdot)$ is a deformation quantization of $(\mathcal{AS}_0(\Sigma), \cdot, \{,\})$, i.e.,
$$\{\alpha, \beta\} = \left(\frac{\alpha\cdot\beta - \beta\cdot\alpha}{h}\right) \quad (mod\ h), \qquad \forall\, \alpha, \beta \in \mathcal{L}.$$

Corresponding results for closed surfaces were first proved by Bullock–Frohman–Kania-Bartoszyńska, Goldman and Przytycki.

Decorated Teichmüller Space and its Weil-Petersson Poisson Structure

- (Penner) The decorated Teichmüller space $\mathcal{T}^d(\Sigma)$ is the space of isotopy classes of complete hyperbolic metrics on Σ with finite area and horocycles associated to the punctures. The arc lengths are defined as distances between horocycles.
- (Mondello) The Weil-Petersson Poisson bi-vector field on $\mathcal{T}^d(\Sigma)$:
$$\Omega_{WP} = \frac{1}{4}\sum_{v\in V}\sum_{\substack{e,e'\in E\\ e\cap e'=v}} \frac{\theta'_v - \theta_v}{r(v)}\frac{\partial}{\partial l(e)}\wedge\frac{\partial}{\partial l(e')}$$
where E is the set of edges of an ideal triangulation of Σ and θ_v and θ'_v respectively are the angles between the e and e' at v.

Relationship between $\mathcal{AS}_0(\Sigma)$ and $\mathcal{T}^d(\Sigma)$

- For an immersed loop or arc α on a decorated hyperbolic surface Σ, let $l(\alpha)$ be the length of the unique geodesic homotopic to α, and let
$$\lambda(\alpha) = \begin{cases} 2\cosh\frac{l(\alpha)}{2} & \text{if } \alpha \text{ is a loop,} \\ e^{\frac{l(\alpha)}{2}} & \text{if } \alpha \text{ is an arc.}\end{cases}$$

Define a map $\Phi : \mathcal{AS}_0(\Sigma) \to C^\infty(\mathcal{T}^d(\Sigma))$ on generators by
$$\alpha \mapsto (-1)^{c(\alpha)}\lambda(\alpha)$$
$$v \mapsto r(v)$$
where $r(v)$ is the length of the horocycle associated to the puncture v, and $c(\alpha)$ is the number of curls that α contains.

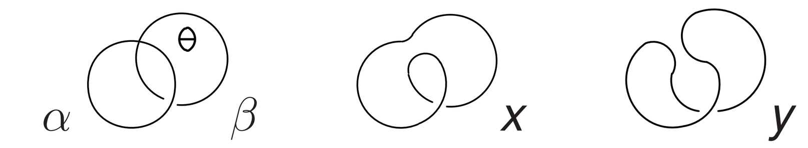
- **Theorem 2** Φ is a well-defined Poisson algebra homomorphism with respect to the generalized Goldman bracket $\{,\}$ on $\mathcal{AS}_0(\Sigma)$ and the Weil-Petersson Poisson structure Ω_{WP} on $\mathcal{T}^d(\Sigma)$.

Compare with Bullock and Przytycki–Sikora for closed surfaces.

- **Corollary** (Generalized Wolpert's Cosine Formula)
$$\Omega_{WP}(l(\alpha), l(\beta)) = \frac{1}{2}\sum_{p\in\alpha\cap\beta\cap\Sigma}\cos\theta_p + \frac{1}{4}\sum_{v\in\alpha\cap\beta\cap V}\frac{\theta'_v-\theta_v}{r(v)} \qquad \forall\, \alpha,\beta\in\mathcal{L}.$$
- **Conjecture** Φ is an injection.

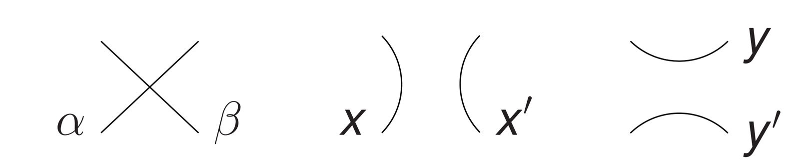
Geodesic Lengths Identities

- (1) For two intersecting closed geodesics α and β (Trace Identity),



$$\cosh\frac{l(x)}{2} + \cosh\frac{l(y)}{2} = 2\cosh\frac{l(\alpha)}{2}\cosh\frac{l(\beta)}{2},$$
$$\cosh\frac{l(x)}{2} - \cosh\frac{l(y)}{2} = 2\sinh\frac{l(\alpha)}{2}\sinh\frac{l(\beta)}{2}\cos\theta.$$

(2) For two intersecting geodesic arcs α and β,



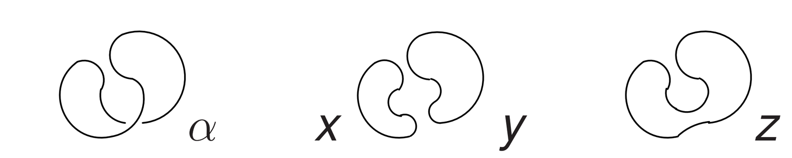
$$e^{\frac{l(x)}{2}}e^{\frac{l(x')}{2}} + e^{\frac{l(y)}{2}}e^{\frac{l(y')}{2}} = e^{\frac{l(\alpha)}{2}}e^{\frac{l(\beta)}{2}},$$
$$e^{\frac{l(x)}{2}}e^{\frac{l(x')}{2}} - e^{\frac{l(y)}{2}}e^{\frac{l(y')}{2}} = e^{\frac{l(\alpha)}{2}}e^{\frac{l(\beta)}{2}}\cos\theta.$$

This is essentially Penner's Ptolemy Relation.

(3) For a geodesic arc α intersecting a closed geodesic β,
$$e^{\frac{l(x)}{2}} + e^{\frac{l(y)}{2}} = 2e^{\frac{l(\alpha)}{2}}\cosh\frac{l(\beta)}{2},$$
$$e^{\frac{l(x)}{2}} - e^{\frac{l(y)}{2}} = 2e^{\frac{l(\alpha)}{2}}\sinh\frac{l(\beta)}{2}\cos\theta.$$

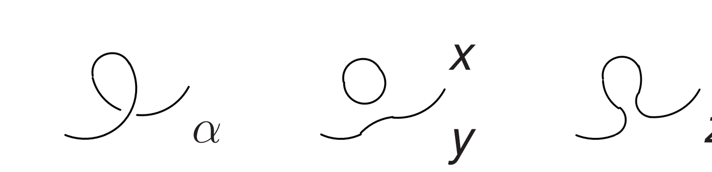
(4) For two geodesic arcs α and β intersecting at a puncture v,
$$e^{\frac{l(x)}{2}} + e^{\frac{l(y)}{2}} = r(v)\, e^{\frac{l(\alpha)}{2}}e^{\frac{l(\beta)}{2}},$$
$$e^{\frac{l(x)}{2}} - e^{\frac{l(y)}{2}} = (\theta'_v - \theta_v)e^{\frac{l(\alpha)}{2}}e^{\frac{l(\beta)}{2}}.$$

(5) For a self-intersecting closed geodesic α,



$$\cosh\frac{l(\alpha)}{2} = 2\cosh\frac{l(x)}{2}\cosh\frac{l(y)}{2} + (-1)^{c(z)}\cosh\frac{l(z)}{2}.$$

(6) For a self-intersecting geodesic arc α,



$$e^{\frac{l(\alpha)}{2}} = 2\cosh\frac{l(x)}{2}e^{\frac{l(y)}{2}} + (-1)^{c(z)}e^{\frac{l(z)}{2}}.$$

(7) For a geodesic arc α self-intersecting at a puncture v,
$$e^{\frac{l(\alpha)}{2}} = \frac{2}{r(v)}\left(\cosh\frac{l(x)}{2} + \cosh\frac{l(y)}{2}\right).$$