Euler equations (beginning of sections 5.4 and problem 34 page 166 of the book)

1. In this section we discuss another class of linear homogeneous equations of second order for which the general solution can be found explicitly, the *Euler equations.* These are equation of the type

$$ax^2y'' + bxy' + cy = 0, \quad x > 0, \tag{1}$$

where a, b, c are real constants, $a \neq 0$

Look for a solution in the form $y(x) = x^\lambda$ for some λ. Substitute to the equation:

$$+\ c \quad \times \quad y(x) = x^\lambda$$
$$+\ bx \quad \times \quad y'(x) = \lambda x^{\lambda-1}$$
$$ax^2 \quad \times \quad y''(x) = \lambda(\lambda-1)x^{\lambda-2}$$

$$ax^2y''(x) + bxy'(x) + cy(x) = (a\lambda(\lambda-1) + b\lambda + c)x^\lambda = 0 \Rightarrow$$
$$\boxed{a\lambda(\lambda-1) + b\lambda + c = 0}$$

So we get the following equation, called the *indicial equation*:

$$\boxed{a\lambda(\lambda-1) + b\lambda + c = 0} \Leftrightarrow \boxed{a\lambda^2 + (b-a)\lambda + c = 0} \tag{2}$$

2. As in the case of equations with constant coefficients the most simple case is the case when the indicial equation has distinct real roots λ_1 and λ_2. In this case the set $\{x^{\lambda_1}, x^{\lambda_2}\}$ forms a fundamental set of solutions and the general solution is

$$y(x) = C_1x^{\lambda_1} + C_2x^{\lambda_2}.$$

EXAMPLE 1. *Find general solution of the equation* $x^2y'' + 2xy' - 6y = 0, \quad x > 0.$

The indicial equation is

$$\lambda(\lambda-1) + 2\lambda - 6 = 0 \Rightarrow$$
$$\lambda^2 + \lambda - 6 = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = -3 \Rightarrow$$

General solution is

$$\boxed{y(x) = C_1x^2 + C_2x^{-3}}$$

3. In order to understand what is the form of general solution for other cases, i.e. when the roots of indicial equation are repeated or complex, we can reduce the equation (1) to the linear equation with constant coefficient by the following change of independent variable:

$$\boxed{x = e^t} \Leftrightarrow \boxed{t = \ln x} \tag{3}$$

Then by chain rule

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = e^t \frac{dy}{dx} = x\frac{dy}{dx} \Leftrightarrow x\frac{dy}{dx} = \frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dx}e^t\right) = \underbrace{\frac{d}{dt}\left(\frac{dy}{dx}\right)e^t}_{\frac{d^2y}{dx^2}(e^t)^2} + \frac{dy}{dx}e^t = x^2\frac{d^2y}{dx^2} + \underbrace{x\frac{dy}{dx}}_{\frac{dy}{dt}} \Rightarrow$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

So the equation (1) is transformed in the new independent variable t to the equation:

$$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0 \tag{4}$$

So, *the indicial equation for the Euler equation* (1) *coincides with the characteristic equation for the equation* (4) and **the general solution for the Euler equation** (1) **is obtained from the general solution to the equation** (4) **by replacing** t **with** $\ln x$ (subsequently $e^{\lambda_i t}$ is replaced by x^{λ_i}).

Using this rule, we can summarize all cases of the roots of the indicial equation in the following table (analogous to the table for the equations with constant coefficients):

SUMMARY:

Solution of the second order Euler equation $ax^2y'' + bxy' + cy = 0, \quad x > 0$. Consider the indicial equation

$$a\lambda(\lambda-1) + br + c = 0 \Leftrightarrow a\lambda^2 + (b-a)\lambda + c = 0$$

and consider its discriminant $D = (b-a)^2 - 4ac$.

Sign of $D = (b-a)^2 - 4ac$	Roots of indicial equation $a\lambda^2 + (b-a)\lambda + c = 0$	General solution
$D > 0$	two distinct real roots $\lambda_1 \neq \lambda_2$	$y(x) = C_1x^{\lambda_1} + C_2x^{\lambda_2}$
$D < 0$	two complex conjugate roots $\lambda_1 = \overline{\lambda_2}$: $\lambda_{1,2} = \alpha \pm i\omega$	$y(x) = x^\alpha\Big(C_1\cos(\omega\ln x) + C_2\sin(\omega\ln x)\Big)$
$D = 0$	two equal(repeated) real roots $\lambda_1 = \lambda_2 = \lambda$	$y(x) = x^\lambda(C_1 + C_2\ln x)$