Other explanations for the case of repeated roots of characteristic equation (section 3.4)

In section 17 we gave a "physical explanation" why the second independent solution for the second order homogeneous equation has a form $te^{\lambda t}$, considering the case of repeated roots as a limiting case of the case of distinct real roots. Now we give two other explanations for this fact.

Factoring differential operator

1. Recall that the **characteristic equation** of a linear homogeneous equation with constant real coefficients
$$ay'' + by' + cy = 0 \tag{1}$$
is
$$a\lambda^2 + b\lambda + c = 0. \tag{2}$$
Assume that
$$D = b^2 - 4ac = 0 \Rightarrow \lambda_1 = \lambda_2 = -\frac{b}{2a}.$$
Set $\lambda := \lambda_1$. So, we found one particular solution $y_1(t) = e^{\lambda t}$.

2. How to find a second particular solution y_2 such that the set $\{y_1, y_2\}$ will be fundamental, i.e. $W(y_1, y_2) \neq 0$?

The method actually works both for distinct and repeated roots.

If λ_1 and λ_2 are roots of the characteristic equation (2), then

Viete formulas

$$a\lambda^2 + b\lambda + c = a(\lambda - \lambda_1)(\lambda - \lambda_2) \Rightarrow \begin{cases} \lambda_1\lambda_2 = \frac{c}{a} \\ \lambda_1 + \lambda_2 = -\frac{b}{a} \end{cases}$$

Then use "factorization" (in Leibnitz notation):

$$ay''+by'+cy = a\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+b\frac{\mathrm{d}y}{\mathrm{d}t}+cy = \left(a\frac{\mathrm{d}^2}{\mathrm{d}t^2} + b\frac{\mathrm{d}}{\mathrm{d}t} + c\right)[y] = a\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_1\right)\left[\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_2\right)[y]\right] = 0$$

Indeed

$$\left(\frac{d}{dt} - \lambda_2\right)[y] = y' - \lambda_2 y$$

$$\left(\frac{d}{dt} - \lambda_1\right)\left[\left(\frac{d}{dt} - \lambda_2\right)[y]\right] = \left(\frac{d}{dt} - \lambda_1\right)(y' - \lambda_2 y) =$$
$$= y'' - \lambda_2 y' - \lambda_1 y' + \lambda_1\lambda_2 y = y'' - (\lambda_1 + \lambda_2)y' + \lambda_1\lambda_2 y =$$
$$\underset{\text{(Viete formulas)}}{=} y'' + \frac{b}{a}y' + \frac{c}{a}.$$

Using this factorization: $ay''+by'+cy=0 \iff$

$a\left(\frac{d}{dt}-\lambda_1\right)\Big(\underbrace{\left[\frac{d}{dt}-\lambda_2\right](y)}_{v}\Big)=0 \Rightarrow \left(\frac{d}{dt}-\lambda_1\right)[v]=0 \iff$

$v'=\lambda_1 v \Rightarrow v=\widetilde{C}_1e^{\lambda_1 t} \Rightarrow \left(\frac{d}{dt}-\lambda_2\right)[y]=\widetilde{C}_1e^{\lambda_1 t} \iff$

$y'-\lambda_2 y=\widetilde{C}_1e^{\lambda_1 t} \Rightarrow$ integrating factor μ satisfies $\mu'=-\lambda_2\mu$

i.e can be taken as $\mu=e^{-\lambda_2 t} \Rightarrow \left(e^{-\lambda_2 t}y\right)'=\widetilde{C}_1e^{(\lambda_1-\lambda_2)t}$

If $\lambda_1\neq\lambda_2 \Rightarrow e^{-\lambda_2 t}y=\underbrace{\frac{\widetilde{C}_1}{\lambda_1-\lambda_2}}_{C_1}e^{(\lambda_1-\lambda_2)t}+C_2 \Rightarrow \boxed{y=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}}$ as we have already seen

3. $\{y_1,y_2\}=\{e^{\lambda t},te^{\lambda t}\}$ is fundamental set. Thus the general solution of (1) is

$$y(t)=C_1e^{\lambda t}+C_2te^{\lambda t}.$$

continuation

If $\lambda_1=\lambda_2=\lambda$ then $\left(e^{-\lambda t}y\right)'=\widetilde{C}_1\cdot e^{\overset{0}{(\lambda-\lambda)}t}=\widetilde{C}_1 \Rightarrow$

$e^{-\lambda t}y=\underset{\substack{\| \\ C_1}}{\widetilde{C}_1}t+C_2 \Rightarrow y=(C_1t+C_2)e^{\lambda t}$

q.e.d.

Reduction of order (part of section 3.4

4. Consider second order linear homogeneous equation with arbitrary coefficients:

$$y''+p(t)y'+q(t)y=0 \tag{3}$$

Assume that we already know one of its particular solutions, $y_1(t)$. How to find another solution to get a fundamental set?

Step 1. Look for a second solution in the form

$$y(t)=v(t)y_1(t).$$

$y(t) = v y_1 \Rightarrow$

$q(t)\times \quad y'(t) = v y_1'(t) + v' y_1$

$+\ p(t) \quad y''(t) = v y_1''(t) + 2v' y_1' + v'' y_1(t)$

$+$

$$y'' + p(t)y'(t) + q(t)y(t) = \Big(\underbrace{y_1''(t) + p(t)y_1'(t) + q(t)y_1}_{0}\Big)v + (2y_1' + p y_1)v' + y_1 v'' = 0$$

Step 2. Set $v' = w$ to reduce order and solve the obtained first order linear homogeneous ODE.

$y_1'' w' + (2y_1' + p y_1) w = 0 \rightarrow$ solve it to find w and then solve $v' = w$ to find v

5. Note that the method can be also applied to linear nonhomogeneous ODE

$$y'' + p(t)y' + q(t)y = g(t).$$

6. Consider the equation with constant coefficients

$$ay'' + by' + cy = 0$$

such that the discriminant $D = b^2 - 4ac$ is equal to zero. Assume that λ is the (repeated) root of the characteristic equation. We now that $y_1(t) = e^{\lambda t}$ is a solution of this equation. Find the second independent solution, using the method of reduction of order.

Look for a solution in the form $y(t) = e^{\lambda t} v \Rightarrow$

$y'(t) = \lambda e^{\lambda t} v + e^{\lambda t} v'$

$y''(t) = \lambda^2 e^{\lambda t} v + 2\lambda e^{\lambda t} v' + e^{\lambda t} v''$

$$\Rightarrow a y'' + by' + cy = \big(\underbrace{a\lambda^2 + b\lambda + c}_{0}\big)e^{\lambda t} v + \big(\underbrace{2a\lambda + b}_{0}\big)e^{\lambda t} v' + e^{\lambda t} v'' = 0 \Leftrightarrow v'' = 0$$

(because λ is a repeated root)

$\rightarrow v = C_2 t + C_1$

More explanation: Note that if λ is a repeated root then $\lambda = -\frac{b}{2a} \Rightarrow$
$2a\lambda + b = 0$ (in fact we use the more general fact that the multiplicity of the root of a polynomial drops by 1 for the derivative of the polynomial)
$\Rightarrow e^{\lambda t} v'' = 0 \Rightarrow v'' = 0 \Rightarrow v = C_2 t + C_1 \Rightarrow$
$y(t) = (C_1 + C_2 t)\, y_1(t) = (C_1 + C_2 t) e^{\lambda t}$ q.e.d.

7.

REMARK 1. *Note that in the case of second order equation we also can use the Abel theorem to find the second independent solution of* (3) *from the knowledge of one solution* $y_1(t)$.

EXPLANATION:

Indeed, by Abel's theorem we know that if $W(t)$ is the Wronskian of y_1 & y_2 then

$$W'(t) + p(t) W(t) = 0 \Rightarrow W = e^{-\int p(t)\,dt} \quad (*)$$

But $$W = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \quad (**)$$

If we know $y_1(t)$ then from $(*)$ and $(**)$ we get the first order equation for y_2.

For example in the case of eq. $ay'' + by' + cy = 0$ with a repeated root λ of char. polynomial: W satisfies

$$W' + \frac{b}{a} W = 0 \Rightarrow W = Ce^{-\frac{b}{a}t}, \quad y_1(t) = e^{\lambda t} \Rightarrow$$

$$\begin{vmatrix} e^{\lambda t} & y_2(t) \\ \lambda e^{\lambda t} & y_2'(t) \end{vmatrix} = Ce^{-\frac{b}{a}t} \Rightarrow e^{\lambda t} y_2'(t) - \lambda e^{\lambda t} y_2(t) = Ce^{-\frac{b}{a}t} \Rightarrow \text{(use that } -\tfrac{b}{a} = 2\lambda)$$

$y_2' - \lambda y_2 = Ce^{\lambda t} \Rightarrow$ by the method of integr. factor $y(t) = (C_1 + C_2 t) e^{\lambda t}$.