

**Repeated Eigenvalues continued:  $n = 3$  with an eigenvalue of algebraic multiplicity 3 (discussed also in problems 18-19, page 437-439 of the book)**

1. We assume that  $3 \times 3$  matrix  $A$  has one eigenvalue  $\lambda_1$  of algebraic multiplicity 3. It means that there is no other eigenvalues and the characteristic polynomial of  $A$  is equal to  $-(\lambda - \lambda_1)^3$ .

Also we have the following three options for geometric multiplicities of  $\lambda_1$ : 1, 2, or 3.

REMARK 1. *If among all  $3 \times 3$  matrices with an eigenvalue with multiplicity 3 one picks up a matrix randomly, then with probability 1 the geometric multiplicity of this eigenvalue will be equal to 1.*

REMARK 2. *For a system of first order linear homogeneous equation that comes from a scalar linear homogeneous equation of higher order, the geometric multiplicity of any eigenvalue is equal to 1. This also shows that the theory of linear systems of first order is more general than the theory of scalar equations of higher order: not every system comes from a single higher order equation.*

2. Actually the considered case can be immediately treated by the Cayley-Hamilton Theorem by analogy with Algorithm 3 for  $n = 2$ : the characteristic polynomial of  $A$  (as a function of variable  $x$ ) is equal to  $-(x - \lambda_1)^3$  so

$$(A - \lambda_1 I)^3 = 0$$

Using this we have that

$$e^{tA} = e^{\lambda_1 t} e^{t(A - \lambda_1 I)} = e^{\lambda_1 t} \left( I + t(A - \lambda_1 I) + \frac{t^2}{2} (A - \lambda_1 I)^2 \right) \quad (1)$$

Then the columns of the resulting matrix form a fundamental set of solutions.

REMARK 3. *As a matter of fact if the geometric multiplicity of  $\lambda_1$  is greater than 1, then the formula (1) can be simplified:*

- *if the geometric multiplicity of  $\lambda_1$  is 3, then  $A = \lambda_1 I$ , i.e.  $A - \lambda_1 I = 0$  so (1) simplifies to*

$$e^{tA} = e^{\lambda_1 t} I; \quad (2)$$

- *if the geometric multiplicity of  $\lambda_1$  is 2, then  $A - \lambda_1 I \neq 0$  but  $(A - \lambda_1 I)^2 = 0$ , so (1) simplifies to*

$$e^{tA} = e^{\lambda_1 t} (I + t(A - \lambda_1 I)); \quad (3)$$

$$(A - \lambda_1 I)w_1 = v$$

- if geometric multiplicity of  $\lambda_1$  is 1, then  $(A - \lambda_1 I)^2 \neq 0$  (but  $(A - \lambda_1 I)^3 = 0$ ), so (1) cannot be simplified.

However, the method of the item above will not work if one has an eigenvalue of algebraic multiplicity 3 for  $n > 3$ , therefore I would like to present the scheme that uses eigenspaces and generalized eigenspaces (i.e. analogous to Algorithms 1 and 2).

3. The case when  $\lambda_1$  has the geometric multiplicity equal to 3 is simple. In this case the eigenspace  $E_{\lambda_1} = \mathbb{R}^3$ , so  $A = \lambda_1 I$  which means that  $A = \lambda_1 I$  and so the formula (2) holds.
4. The case when  $\lambda_1$  has the geometric multiplicity equal to 2:
  - In this case  $\dim E_{\lambda_1} = 2$
  - Then  $\dim E_{\lambda_1}^{(2)} = 3$
5. Therefore we have the following analog of Algorithm 1:

- (a) Suppose we found  $E_{\lambda_1}$  and it is 2-dimensional. Then take any  $w$  which is not in  $E_{\lambda_1}$ . Then  $w$  is a generalized eigenvector of order 2. Set

$$v_1 = (A - \lambda_1 I)w$$

Then  $v_1$  is a generalized eigenvector of order 1, i.e.  $v_1$  is an eigenvector of  $\lambda_1$ .

- (b) Choose another eigenvector  $v_2$  of  $\lambda_1$  which is not collinear to  $v_1$ , so that  $v_1$  and  $v_2$  form a basis of the eigenspace  $E_{\lambda_1}$ .
- (c) Then  $\{v_1, w, v_2\}$  form a basis of generalized eigenvectors in  $\mathbb{R}^3$  (with  $v_1$  and  $v_2$  being eigenvectors and  $w$  being a generalized eigenvector of order 2) and so

$$(e^{tA}v_1, e^{tA}w, e^{tA}v_2) = \{e^{\lambda_1 t}v_1, e^{\lambda_1 t}(w + tv_1), e^{\lambda_1 t}v_2\}$$

form a fundamental set of solutions of  $X' = AX$ .

6. You can try to build an analog of Algorithm 2 here (see also discussions in Problem 19 on page 438-439 of the book): First find the eigenspace  $E_{\lambda_1}$ , which is a plane. The problem here that, if we choose some eigenvector  $v$ , the system

$$(A - \lambda_1 I)w = v \tag{4}$$

does not have necessarily a solution, so you need to choose a specific  $v$  in  $E_{\lambda_1}$  for which (4) has a solution (see item (c) of Problem 19, page 439). **In contrast, Algorithm 1 does not have this problem at all, so it is certainly more efficient in this and more general situations than the Algorithm 2.**

7. The case when  $\lambda_1$  has the geometric multiplicity equal to 1:

- In this case  $\dim E_{\lambda_1} = 1$ ;
- Then  $\dim E_{\lambda_1}^{(2)} = 2$  (it can be actually shown that  $\dim E_{\lambda_1}^{(k+1)} - \dim E_{\lambda_1}^{(k)} \leq \dim E_{\lambda_1}^{(k)} - \dim E_{\lambda_1}^{(k-1)}$  and, in particular,  $\dim E_{\lambda_1}^{(2)} - \dim E_{\lambda_1}^{(1)} \leq \dim E_{\lambda_1}^{(1)}$ );
- Then  $\dim E_{\lambda_1}^{(3)} = 3$ , so  $E_{\lambda_1}^{(3)} = \mathbb{R}^3$ .

8. Analog of Algorithm 1:

- (a) Suppose we already found the spaces  $E_{\lambda_1}$  (by solving the system  $(A - \lambda_1 I)v = 0$ ) and it is 1-dimensional as expected. Then we find the space  $E_{\lambda_1}^2$  by solving the system  $(A - \lambda_1 I)^2 w = 0$ .  $E_{\lambda_1}^2$  must be 2-dimensional;
- (b) Choose any vector  $w_2$  which is not in the plane  $E_{\lambda_1}^2$ . Then  $w_2$  is the generalized eigenvector of order 3. Then set

$$w_1 := (A - \lambda_1 I)w_2, \quad v = (A - \lambda_1 I)w_1$$

- (c) Then  $w_1$  is a generalized eigenvector of order 2 and  $v$  is a generalized eigenvector of order 1. Also, vectors  $\{v, w_1, w_2\}$  form a basis of  $\mathbb{R}^3$  and

$$e^{tA}v = e^{\lambda_1 t}v; \tag{5}$$

$$e^{tA}w_1 = e^{\lambda_1 t}(w_1 + tv); \tag{6}$$

$$e^{tA}w_2 = e^{\lambda_1 t}\left(w_2 + tw_1 + \frac{t^2}{2}v\right) \tag{7}$$

So,

$$\{e^{\lambda_1 t}v, e^{\lambda_1 t}(w_1 + tv), e^{\lambda_1 t}\left(w_2 + tw_1 + \frac{t^2}{2}v\right)\} \tag{8}$$

form a fundamental set of solutions of the system  $X' = AX$ .

**REMARK 4.** Note that if in the right-hand sides of equations (5)-(7) one removes the factor  $e^{\lambda_1 t}$  the remaining expression is the derivative of the next one. This is not occasional and can be generalized to the cases of higher multiplicities.

9. Analog of Algorithm 2:

- (a) Choose an eigenvector  $v$  (all eigenvectors are collinear one to each other);
- (b) Find  $w_1$  such that  $(A - \lambda_1 I)w_1 = v$ ;
- (c) Find  $w_2$  such that  $(A - \lambda_1 I)w_2 = w_1$ ;
- (d) Proceed as in the item (c) of the previous algorithm.

REMARK 5. *In general the geometric and algebraic multiplicity are not the only characteristics of an eigenvalue as in the small dimensional examples we discussed. In higher dimensions there are additional characteristics related to partitions of an algebraic multiplicities into the sum of positive integers, then the geometric multiplicity is the number of terms in the partition. This is related to the theory of elementary divisors. In general the topic of repeated eigenvalue is related to the topic of **Jordan normal forms** in Linear Algebra. Actually we implicitly learnt how to construct the Jordan normal form at least for  $n \leq 3$ , but the general scheme should be clear for attentive reader.*