12 Vectors and Geometry of Space

12.1: Three-dimensional Coordinate System

The three-dimensional coordinate system consists of the **origin** O and the **coordinate axes**: x-axis, y-axis, z-axis.The coordinate axes determine 3 **coordinate planes**: the xy-plane, the xz-plane and yz-plane. The coordinate planes divide space into 8 parts, called octants.

Representation of point $P(a, b, c)$ and its projections on the coordinate planes:

EXAMPLE 1. *Describe in words the regions of* $\mathbb{R}^3$ *represented by the following equation:*

(a) $z = 0$

(b) $y = 0$

(c) $x = 0$

Note that in $\mathbb{R}^2$ the graph of the equation involving x and y is a curve. In $\mathbb{R}^3$ an equation in x, y, z represents a **surface**.(It does not mean that we can't graph curves in $\mathbb{R}^3$.)

EXAMPLE 2. *Sketch the graph of* $x^2 + y^2 - 1 = 0$ *in* $\mathbb{R}^2, \mathbb{R}^3$.

A **cylinder** is a surface of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.

An equation that contains only two of the variables x, y, z *represents a* **cylindrical surface** *in* $\mathbb{R}^3$. *How to graph cylindrical surface:*

1. *graph the equation in the coordinate plane of the two variables that appear in the given equation;*
2. *translate that graph parallel to the axis of the missing variable.*

EXAMPLE 3. *Sketch the graph of* $(x + 2)^2 + (y - 4)^2 = 1$ *in* $\mathbb{R}^3$

EXAMPLE 4. *Sketch the graph of* $y = x^2$ *in* $\mathbb{R}^3$

EXAMPLE 5. *Let S be the graph of $x^2 + z^2 - 10z + 21 = 0$ in $\mathbb{R}^3$.*

(a) *Describe S.*

(b) *The intersection of S with the xz- plane is*____

(c) *The intersection of S with the yz- plane is*____

(d) *The intersection of S with the xy- plane is*____

Spheres

- **Distance formula in $\mathbb{R}^3$**: The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

EXAMPLE 6. *Show that the equation $x^2 + y^2 + z^2 + 2x - 4y + 8z + 17 = 0$ represents a sphere, and find its center and radius.*

In general, completing the squares in

$$x^2 + y^2 + z^2 + Gx + Hy + Iz + J = 0$$

produces an equation of the form

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = k$$

- If $k > 0$ then the graph of this equation is ______________________________
- If $k = 0$, then the graph is ______________________________
- If $k < 0$ then ______________________________

Regions in $\mathbb{R}^3$

EXAMPLE 7. *Describe the set of all points in $\mathbb{R}^3$ whose coordinates satisfy the following inequality:* $x^2 + y^2 < 16$

EXAMPLE 8. *Describe the following region:*$\{(x, y, z) | 9 \leq x^2 + y^2 + z^2 \leq 16\}$