12.5: Equations of lines and planes

Lines

Lines determined by a point and a vector

Consider line L that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the nonzero vector $\mathbf{v} = \langle a, b, c\rangle$.

Parametric equations of the line:

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}$$

EXAMPLE 1. *Find parametric equations of the line*

(a) *passing through the point* $(3, -4, 1)$ *and parallel to* $\mathbf{v} = \langle 7, 0, -1\rangle$

(b) *passing through the origin and parallel to* $\mathbf{v} = \langle 5, 5, 5\rangle$

EXAMPLE 2. *Consider the line* L *that passes through the points* $A(1, 1, 1)$ *and* $B(2, 3, -2)$*. Find points at that* L *intersects the* yz*-plane.*

Symmetric equations of the line: If $abc \neq 0$ then

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

If, for example, $a = 0$ then the symmetric equations have the form:

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

EXAMPLE 3. *Find symmetic equations of lines from Example 1.*

Vector equation of the line:

$$\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$$

where $P_0(x_0, y_0, z_0)$ is a given point on the line and $\mathbf{v} = \langle a, b, c \rangle$ is some vector which is parallel to the line, t is a parameter, $-\infty < t < \infty$.

EXAMPLE 4. *Find vector equation of the line that passes through the points $P(1, 1, -4)$ and $Q(0, 3, -4)$.*

EXAMPLE 5. *Determine whether the lines*

$$L_1: \quad x-1=\frac{y+2}{3}=\frac{z-4}{-1}$$

and

$$L_2: \quad x=2t, \quad y=3+t, \quad z=-3+4t$$

are parallel, skew, or intersecting.

Summarizing table

Vector equation	Parametric equations	Symmetric equations

$\mathbf{r}(t)=\langle x_0,y_0,z_0\rangle+t\langle a,b,c\rangle$,
 $-\infty<t<\infty$
	$x = x_0+at$,
	 $y = y_0+bt$,
	 $z = z_0+ct$,
	 $-\infty<t<\infty$
		If $abc\neq 0$ then
		 $\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$
		 If, for example, $a=0$ then
		 the symmetric equations have the form:
		 $x=x_0, \quad \frac{y-y_0}{b}=\frac{z-z_0}{c}$

Line segments

How to find parametric equation of a line segment:

1. *Find parametric equation for the entire line;*

2. *restrict the parameter appropriately so that only the desired segment is generated.*

EXAMPLE 6. *Find parametric equations describing the line segment joining the points* $M(1, 2, 3)$ *and* $N(3, 2, 1)$.

Planes

Planes parallel to the coordinate planes:

Planes determined by a point and a normal vector

A plane in $\mathbb{R}^3$ is uniquely determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n} = (a, b, c)$ that is orthogonal to the plane. This vector is called a **normal vector**.

Assume that $P(x, y, z)$ is any point in the plane. Let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors for P_0 and P respectively.

Vector equation of the plane: $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}.$

Scalar equation of plane:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Often this will be written as a **linear equation** in x, y, z,

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

EXAMPLE 7. *Determine the equation of the plane through the point* $(1, 2, 1)$ *and orthogonal to vector* $\langle 2, 3, 4\rangle$. *Find the intercepts and sketch the plane.*

EXAMPLE 8. *Determine the equation of the plane through the points* $A(1, 1, 1)$, $B(0, 1, 0)$ *and* $C(1, 2, 3)$.

Two planes are **parallel** if their normal vectors are parallel.

Two planes are **orthogonal** if their normal vectors are orthogonal.

If two planes are not parallel, then they intersect in a straight line and the **angle** between the two planes is defined as the *acute* angle between their normal vectors.

EXAMPLE 9. *Given four planes:*

$$\begin{array}{lrcrcrcrcl} P_1: & 2x & + & 3y & + & z & + & 11 & = & 0 \\ P_2: & -4x & - & 6y & - & 2z & + & 77 & = & 0 \\ P_3: & 2x & & & - & 4z & + & 33 & = & 0 \\ P_4: & -2x & + & 3y & + & z & + & 11 & = & 0. \end{array}$$

Determine whether the given pairs of the planes are parallel, orthogonal, or neither. Find the angle between the planes.

(a) P_1 *and* P_2

(b) P_1 *and* P_3

(c) P_2 *and* P_3

(d) P_1 *and* P_4

Line as an intersection of two non parallel planes:

$$L: \begin{cases} a_1x + b_1y + c_1z + d_1 = 0 \\ a_2x + b_2y + c_2z + d_2 = 0 \end{cases}$$

The direction vector of L is $\mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2$.

EXAMPLE 10. *Find an equation of the line given as intersection of two planes:*

$$\begin{array}{rcrcrcl} x & - & y & + & 3z & = & 0 \\ x & + & y & + & 4z & = & 2 \end{array}$$