13.3: Arc Length and curvature of a curve

The length of a curve

The length of a plane curve with parametric equations $x = x(t)$, $y = y(t)$, $a \le t \le b$, as the limit of lengths of inscribed broken lines is

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

under the assumption that $x'(t)$ and $y'(t)$ exist and continuous

In exactly the same way the length of a space curve given by the parametric equation

$$x = x(t), y = y(t), z = z(t), \quad a \le t \le b,$$

as the limit of lengths of inscribed broken lines, is

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

under the assumption that $x'(t)$, $y'(t)$, and $z'(t)$ exist and continuous.

Both cases of lengths of plane and space curves can be simultaneously written as

$$L = \int_a^b |\mathbf{r}'(t)|\, dt$$

where

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

is the vector function describing the curve.

EXAMPLE 1. *Find the length of the curve:*

(a) $\mathbf{r}(t) = \langle -2t, 5\cos t, 5\sin t\rangle, \quad -10 < t < 10.$

(b) $\mathbf{r}(t) = \langle 4t, \frac{2\sqrt{2}}{3}t^3, \frac{1}{5}t^5 \rangle, \quad 0 \le t \le 2$

The Arc Length Function/Parametrization

Assume again that a curve C is given by the vector function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \le t \le b.$$

The function

$$s(t) = \int_a^t |\mathbf{r}'(\tau)|\, d\tau = \int_a^t \sqrt{x'(\tau)^2 + y'(\tau)^2 + z'(\tau)^2}\, d\tau \;\Leftrightarrow\; \frac{ds}{dt} = |\mathbf{r}'(t)|,\; s(a) = 0.$$

is called the *arc length function* of the curve C.

The geometric meaning: $s(t)$ is the length of the part of C between the points $\mathbf{r}(a)$ and $\mathbf{r}(t)$.

If L is the Length of the curve C then for every $s \in [0, L]$ there exists unique $t(s)$ such that the length of the curve between $\mathbf{r}(a)$ and $\mathbf{r}\big(t(s)\big)$ is equal to s.

The curve given by the vector function

$$\widetilde{\mathbf{r}}(s) := \mathbf{r}\big(t(s)\big)$$

trace out the same curve C in $\mathbb{R}^3$, as $\mathbf{r}(t)$, but the motion along C with respect to the new parameter s is different , than with respect to the old parameter t. More precisely the motion along C with respect to s has the unit speed at every points, because

$$\left|\frac{d\widetilde{\mathbf{r}}}{ds}\right| = \left|\frac{d\mathbf{r}}{dt}\right| \left|\frac{dt}{ds}\right| = 1.$$

By finding the function $t = t(s)$ (i.e., the inverse to the arc length function), one *reparametrizes the curve C with respect to the arc length*

EXAMPLE 2. *Reparametrize the curve with respect to arc length measured from the point where $t = 0$ in the direction of increasing t, i.e. find $t(s)$, if*
$\mathbf{r}(t) = (1 + 4t)\mathbf{i} + (1 - 3t)\mathbf{j} + (1 + 2t)\mathbf{k}$

Curvature of the curve

The *unit tangent vector*

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Assume that the curve is parametrized by arc length parameter s, i.e. is represented by a vector function $\mathbf{r}(s)$ such that $|r'(s)| = 1$. Then the curvature $k(s)$ (at the point $\mathbf{r}(s)$ is defined as follows

$$k(s) = |r''(s)| = \left|\frac{d\mathbf{T}}{ds}\right|,$$

i.e., the curvature of the curve is the magnitude of the acceleration if one moves along the curve with unit speed.

If a curve C is parametrized by a parameter T (which not necessary by the arc length parametriation), then

$$k(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \tag{1}$$

EXAMPLE 3. *What is the curvature of the circle of radius R*

REMARK 4. The curvature of a curve is identically equal to zero if and only if the curve is a straight line

EXAMPLE 5. *Let* $\mathbf{r}(t) = \langle 3\cos 2t, 3\sin 2t, -3t\rangle$, $t \in \mathbb{R}$.

(a) *Find the unit tangent* $T(t)$ *of the curve given by* $\mathbf{r}(t)$

(b) *Find the curvature of the curve given by* $\mathbf{r}(t)$.

REMARK 6. From (1), using the fact that given two vctors $\mathbf{a}$ and $\mathbf{b}$ one has $\sqrt{|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2} = |\mathbf{a}\times\mathbf{b}|$, one can deduce the following formula for the curvature in terms of $\mathbf{r}(\mathbf{t})$ and their derivatives up to order 2:

$$k(T) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Prncipal unit normal vector

Note that $|\mathbf{T}|^2 = \mathbf{T}(t)\cdot\mathbf{T}(t) = 1$. Differentiating this identity we get that $2\mathbf{T}'(t)\cdot\mathbf{T}(t) = 0$, i.e $\mathbf{T}'(t) \perp \mathbf{T}(t)$. Assume that $\mathbf{T}'(t) \neq 0$ (equivalently the curvature $k(t) \neq 0$). Normalizing $T'(t)$ we obtain so called *principal unit normal vector* or simply *unit normal*:

$$\mathbf{N}(t) := \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

Geometric meaning of the unit normal: the direction of the acceleration if one moves with along the curve with the unit speed.

EXAMPLE 7. *Find the unit normal for the curve from Example 5.*

REMARK 8 (The notions discussed here will not be used in homework or tests, it is just an extra material). The vector

$$\mathbf{B}(T) := \mathbf{T}(t) \times \mathbf{N}(t)$$

is called the *binormal vector*. Using the binormal vector one can define another function $\tau(t)$ on the curve called *torsion*. In the arc length parameter s

$$|\tau(s)| = \left| \frac{d\mathbf{B}(s)}{ds} \right|.$$

(the sign of $\tau(s)$ can be also specified but at this stage it is not important) In the arbitrary parameter T

$$|\tau(t)| = \frac{|B'(t)|}{|\mathbf{r}'(t)|}$$

The triple of vectors $\Big(\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\Big)$ is called the *Frenet-Serret moving frame* and it is the basic tool in the study of curves in $\mathbb{R}^3$ up to the rigid motions, i.e. translations and rotations. Using the Frenet-Serres it can be shown that the curvature and the torsion define the curve in $\mathbb{R}^3$ uniquely up to a rigid motion.