14.1: Functions of Several Variables

Consider the following formulas:

$$z = 2 - x - 4y \tag{1}$$

$$z = x^2 + y^2 \tag{2}$$

$$z = \sqrt{x^2 + y^2} \tag{3}$$

$$z = \sqrt{1 - x^2 - y^2} \tag{4}$$

$$\boxed{\begin{array}{c}\text{ordered pair of independent variables } (x, y)\\ \text{(input)}\end{array}} \longrightarrow \boxed{} \longrightarrow \boxed{\begin{array}{c}\text{dependent variable } z\\ \text{(output)}\end{array}}$$

DEFINITION 1. *Let $D \subset \mathbb{R}^2$. A* **function** f **of two variables** *is a rule that assigns to each ordered pair (x, y) in D a unique real number denoted by $f(x, y)$.*

The set D is the **domain** *of f and the* **range** *of f is the set of values that f takes on, that is $\{f(x, y) | (x, y) \in D\}$.*

REMARK 2. Obviously, one can choose the independent variables arbitrary, for example, $x = f(y, z)$.

- **GRAPH of** $f(x, y)$.

Recall that a graph of a function f of one variable is a curve C with equation $y = f(x)$.

DEFINITION 3. *The* **graph** *of f with domain D is the set:*

$$S = \left\{(x, y, z) \in \mathbb{R}^3 | z = f(x, y). \;\; (x, y) \in D\right\}.$$

The graph of a function f of two variables is a surface S in three dimensional space with equation $z = f(x, y)$.

EXAMPLE 4. *Find the domain and sketch the graph of the functions (1)-(4). What is the range?*

(1) $z = 2 - x - 4y$
$D =$

(2) $z = x^2 + y^2$
$D =$

(3) $z = \sqrt{x^2 + y^2}$
$D =$

(4) $z = \sqrt{1 - x^2 - y^2}$
$D =$

EXAMPLE 5. *Sketch the domain of each of the following:*

(a) $z = \sqrt{x} - \dfrac{5}{\sqrt{y}}$

(b) $z = \ln(x^2 + \dfrac{y^2}{16} - 1)$

• **LEVEL (CONTOUR) CURVES** method of visualizing functions is the method borrowed from mapmakers. It is a contour map on which points of constant elevation are joined to form level (or contour) curves.

DEFINITION 6. *The* **level (contour) curves** *of a function of two variables are the curves with equations*

$$f(x, y) = k,$$

where k is a constant in the range of f.

A level curve is the locus of all points at which f takes a given value k (it shows where the graph of f has height k).

EXAMPLE 7. *Sketch the level curves of the functions (2) and (3) for the values* $k = 0, 1, 2, 3, 4$*:*

(2) $z = x^2 + y^2$ (3) $z = \sqrt{x^2 + y^2}$

- **Functions of three variables.**

DEFINITION 8. *Let* $D \subset \mathbb{R}^3$*. A* **function** f **of three variables** *is a rule that assigns to each ordered pair* (x, y, z) *in* D *a unique real number denoted by* $f(x, y, z)$*.*

Examples of functions of 3 variables:

$$f(x, y, z) = x^2 + y^2 + z^2,$$

$$u = xyz$$

$$T(s_1, s_2, s_3) = \ln s_1 + 12s_2 - s_3^{-5}.$$

Emphasize that domains of functions of three variables are regions in three dimensional space.

EXAMPLE 9. *Find the domain of the following function:*

$$f(x, y, z) = \frac{\ln(36 - x^2 - y^2 - z^2)}{\sqrt{x^2 + y^2 + z^2 - 25}}.$$

Note that for functions of three variables it is impossible to visualize its graph. However we can examine them by their **level surfaces**:

$$f(x, y, z) = k$$

where k is a constant in the range of f. If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

EXAMPLE 10. *Find the level surfaces of the function* $u = x^2 + y^2 - z$.

REMARK 11. For any function there exist a unique level curve (surface) through given point!!!