14.5: The Chain Rule

Chain Rule for functions of a single variable: If $y = f(x)$ and $x = g(t)$ where f and g are differentiable functions, then y is indirectly a differentiable function of t and

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}g}{\mathrm{d}t}.$$

EXAMPLE 1. *Let $z = x^y$, where $x = t^2$, $y = \sin t$. Compute $z'(t)$.*

`Assume that all functions below have continuous derivatives (ordinary or partial).`

- CASE 1: $z = f(x, y)$, where $x = x(t)$, $y = y(t)$ and compute $z'(t)$.

  Chain Rule:
  $$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

  SOLUTION OF EXAMPLE 1:

EXAMPLE 2. *The radius of a right circular cone is increasing at a rate of* 1.8 *cm/s while its height is decreasing at a rate* 2.5 *cm/s. At what rate is the volume of the cone changing when the radius is* 120 *cm and the height is* 140 *cm.*

- CASE 2: $z = f(x, y)$, where $x = x(s, t)$, $y = y(s, t)$ and compute z_s and z_t.

  Chain Rule: *Tree diagram:*

  $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

  $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

EXAMPLE 3. *Write out the Chain Rule for the case where* $w = f(x, y, z)$ *and* $x = x(u, v)$, $y = y(u, v)$ *and* $z = z(u, v)$.

EXAMPLE 4. *If* $z = \sin x \cos y$, *where* $x = (s-t)^2$, $y = s^2 - t^2$ *find* $z_s + z_t$.

EXAMPLE 5. *Show that*

$$g(s,t) = f(s^2 - t^2, t^2 - s^2)$$

satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0.$$

EXAMPLE 6. *If* $u = x^2y + y^3z^2$ *where* $x = rse^t$, $y = r + s^2e^{-t}$, $z = rs\sin t$, *find* u_s *when* $(r, s, t) = (1, 2, 0)$

Implicit differentiation: Suppose that an equation

$$F(x, y) = 0$$

defines y implicitly as a differentiable function of x, i.e. $y = y(x)$, where $F(x, y(x)) = 0$ for all x in the domain of $y(x)$. Find y':

EXAMPLE 7. *Find y' if $x^4 + y^3 = 6e^{xy}$.*

Suppose that an equation

$$F(x, y, z) = 0$$

defines z implicitly as a differentiable function of x and y, i.e. $z = z(x, y)$, where

$$F(x, y, z(x, y)) = 0$$

for all (x, y) in the domain of z. Find the partial derivatives z_x and z_y:

EXAMPLE 8. *If* $x^4 + y^3 + z^2 + xye^z = 10$ *find*

(a) z_x *and* z_y

(b) x_y *and* x_z