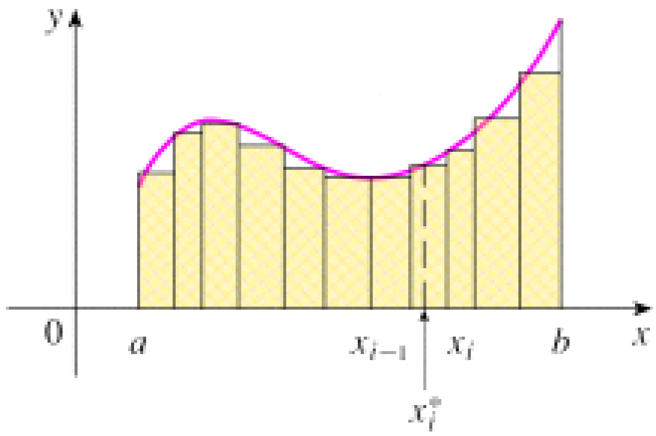
15.1: Double integrals over rectangles

Recall that a single definite integral can be interpreted as **area**:

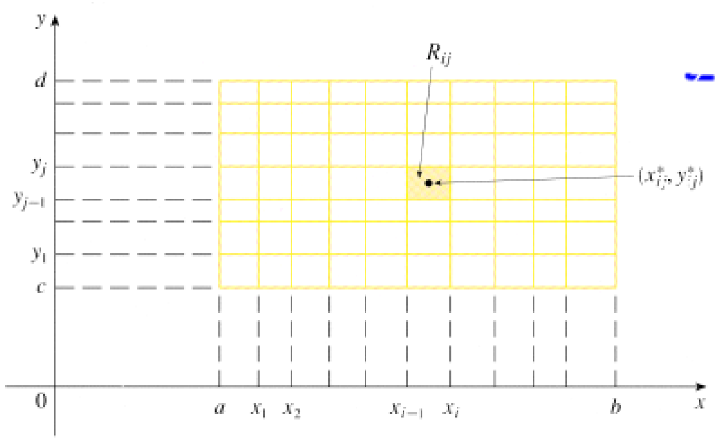


The exact area is also the definition of the definite integral:

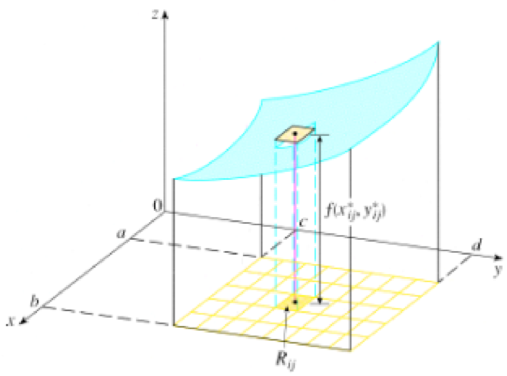
$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

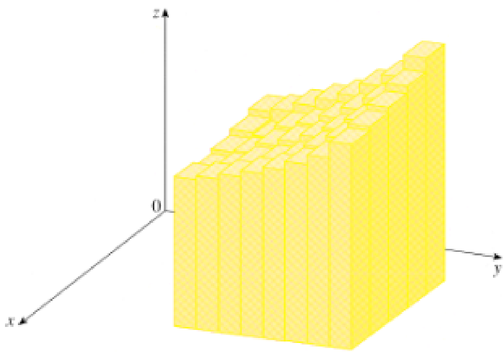
Problem: Assume that $f(x, y)$ is defined on a closed rectangle $R = [a, b] \times [b, c] = \{(x, y) \in \mathbb{R}^2 | a \le x \le b, c \le y \le d\}$ and $f(x, y) \ge 0$ over R. Denote by S the part of the surface $z = f(x, y)$ over the rectangle R. What the volume of the region under S and above the xy-plane is?

Solution: Approximate the volume. Divide up $a \le x \le b$ into n subintervals and divide up $c \le y \le d$ into m subintervals. From each of these smaller rectangles choose a point (x_i^*, y_j^*).



Over each of these smaller rectangles we will construct a box whose height is given by $f(x_i*, y_j^*)$.





The volume is given by

$$\lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_j^*)\Delta x \Delta y$$

which is also the definition of a double integral

$$\iint_R f(x, y)\mathrm{d}A.$$

Another notation: $\iint_R f(x, y)\,\mathrm{d}A = \iint_R f(x, y)\,\mathrm{d}x\mathrm{d}y.$

THEOREM 1. *If f is continuous on R then f is integrable over R.*

THEOREM 2. *If $f(x, y) \geq 0$ and f is continuous on the rectangle $R = [a, b] \times [c, d]$, then the volume V of the solid S that lies above R and inder the graph of f, i.e.*

$$S = \left\{(x, y, z) \in \mathbb{R}^3 | (x, y) \in R, 0 \leq z \leq f(x, y), (x, y) \in R\right\},$$

is

$$V = \iint_R f(x, y)\,\mathrm{d}A.$$

EXAMPLE 3. *Evaluate the integral*

$$\iint_R 4\,\mathrm{d}A$$

where $R = [-1, 0] \times [-3, 3]$ by identifying it as a volume of a solid.

Iterated integrals

Suppose that $f(x, y)$ is integrable over the rectangle $R = [a, b] \times [b, c]$.

Partial integration of f with respect to x: $\displaystyle\int_a^b f(x, y)\,\mathrm{d}x$

Partial integration of f with respect to y: $\displaystyle\int_c^d f(x, y)\,\mathrm{d}y$

EXAMPLE 4.

$$\int_0^4 (x + 3y^2)\,\mathrm{d}x =$$

$$\int_1^4 e^{xy}\,\mathrm{d}y =$$

Iterated integrals:

$$\int_a^b \left[\int_c^d f(x, y)\,\mathrm{d}y\right]\mathrm{d}x = \int_a^b \int_c^d f(x, y)\,\mathrm{d}y\mathrm{d}x$$

and

$$\int_c^d \left[\int_a^b f(x, y)\,\mathrm{d}x\right]\mathrm{d}y = \int_c^d \int_a^b f(x, y)\,\mathrm{d}x\mathrm{d}y.$$

EXAMPLE 5. *Evaluate the integrals:*

$$I_1 = \int_0^{\ln 2} \int_0^{\ln 5} e^{2x-y}\,\mathrm{d}y\mathrm{d}x, \quad I_2 = \int_0^{\ln 5} \int_0^{\ln 2} e^{2x-y}\,\mathrm{d}x\mathrm{d}y$$

FUBINI's THEOREM: *If f is continuous on on the rectangle $R = [a, b] \times [c, d]$ then*

$$\iint_R f(x, y)\,\mathrm{d}A = \int_a^b \int_c^d f(x, y)\,\mathrm{d}y\mathrm{d}x = \int_c^d \int_a^b f(x, y)\,\mathrm{d}x\mathrm{d}y.$$

EXAMPLE 6. *Make a conclusion from the Example 5 based on the Fubini Theorem.*

COROLLARY 7. *If g and h are continuous functions of one variable and $R = [a, b] \times [c, d]$ then*

$$\iint_R g(x)h(y)\,\mathrm{d}A = \left(\int_a^b g(x)\mathrm{d}x\right)\left(\int_c^d h(y)\,\mathrm{d}y\right).$$

EXAMPLE 8. *Evaluate*

$$\iint_R x\cos(xy)\,\mathrm{d}A$$

where $R = [-\pi/2, \pi/2] \times [1, 5]$ and describe your result geometrically.

EXAMPLE 9. *Express the volume of the solid S lying under the circular paraboloid $z = x^2 + y^2$ and above the rectangle $R = [-2, 2] \times [-3, 3]$ using an iterated integral.*