15.2: Double Integrals over General Regions

`All functions below are continuous on their domains.`

Let D be a bounded region enclosed in a rectangular region R such that its boundary ∂D is sufficiently nice, for example, is a piecewise differentiable curve. We define

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D. \end{cases}$$

If F is integrable over R, then we say F is *integrable* over D and we define **the double integral of f over D** by

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_R F(x,y)\,\mathrm{d}A$$

FACT: If $f(x,y) \geq 0$ and f is continuous on the region D then the volume V of the solid S that lies above D and under the graph of f, i.e.

$$S = \left\{(x,y,z) \in \mathbb{R}^3 |\ 0 \leq z \leq f(x,y), (x,y) \in D\right\},$$

is

$$V = \iint_D f(x,y)\,\mathrm{d}A.$$

EXAMPLE 1. *Evaluate the integral*

$$\iint_D \sqrt{16 - x^2 - y^2}\,\mathrm{d}A$$

where $D = \left\{(x,y) \in \mathbb{R}^2 |\ x^2 + y^2 \leq 16\right\}$ by identifying it as a volume of a solid.

Properties of double integrals:

- If $D = D_1 \cup D_2$, where D_1 and D_2 do not overlap except perhaps their boundaries then

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D_1} f(x,y)\,\mathrm{d}A + \iint_{D_2} f(x,y)\,\mathrm{d}A.$$

- If α and β are real numbers then

$$\iint_D (\alpha f(x,y) + \beta g(x,y))\,\mathrm{d}A = \alpha \iint_D f(x,y)\,\mathrm{d}A + \beta \iint_D g(x,y)\,\mathrm{d}A.$$

- If we integrate the constant function $f(x,y) = 1$ over D, we get **area** of D:

$$\iint_D 1\,\mathrm{d}A = A(D).$$

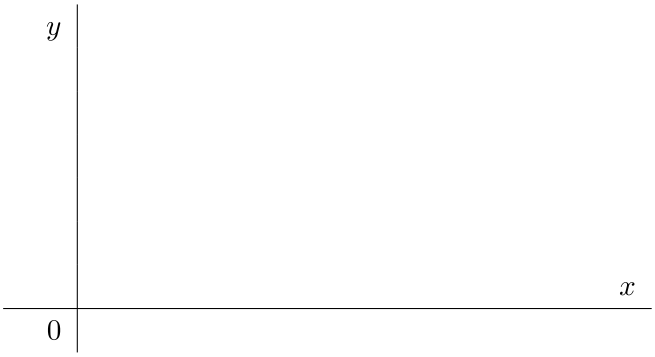
EXAMPLE 2. *If* $D = \{(x,y) \mid x^2 + y^2 \le 25\}$ *then*

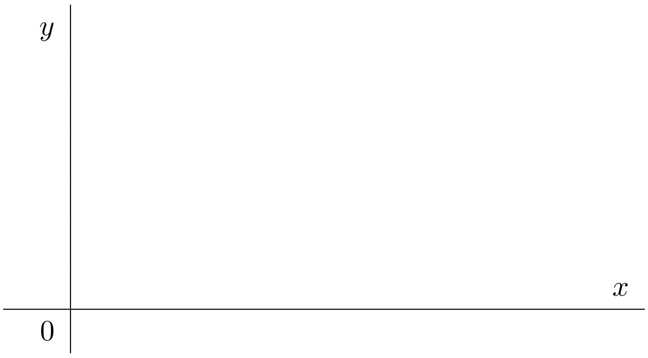
$$\iint_D \mathrm{d}A =$$

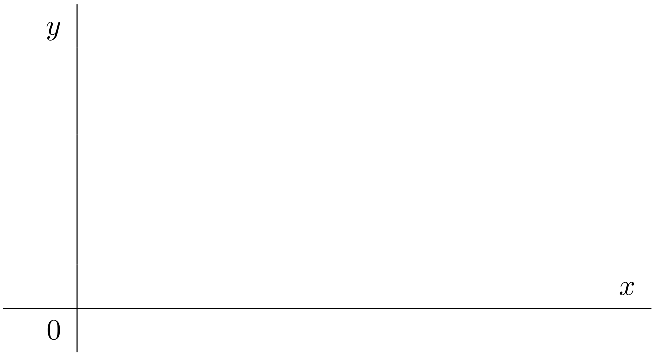
Computation of double integral:

A plain region of **TYPE I**:

$$D = \{(x, y) |\ a \le x \le b, g_1(x) \le y \le g_2(x)\}.$$





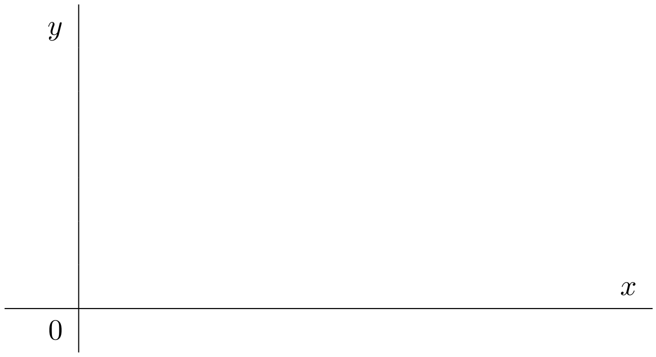


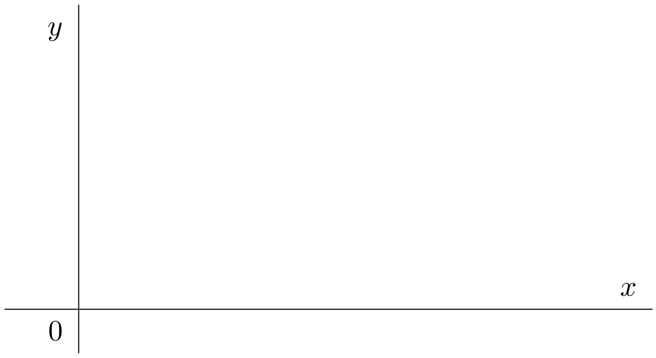
THEOREM 3. *If D is a region of type I such that $D = \{(x, y) |\ a \le x \le b, g_1(x) \le y \le g_2(x)\}$ then*

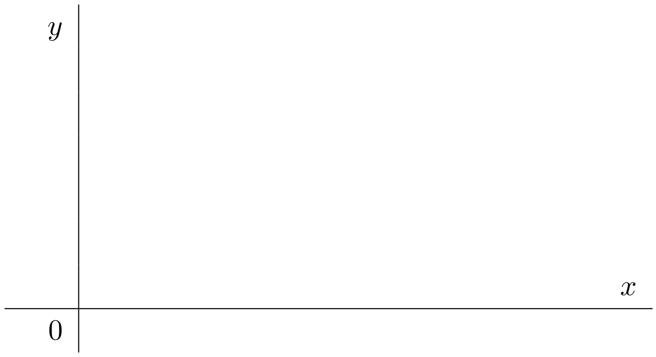
$$\iint_D f(x, y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,\mathrm{d}y\mathrm{d}x.$$

A plain region of **TYPE II**:

$$D = \{(x, y) |\ c \le y \le d, h_1(y) \le x \le h_2(y)\}.$$







THEOREM 4. *If D is a region of type II s.t. $D = \{(x, y) |\ c \le y \le d, h_1(y) \le x \le h_2(y)\}$ then*

$$\iint_D f(x, y)\,\mathrm{d}A = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,\mathrm{d}x\mathrm{d}y.$$

EXAMPLE 5. *Evaluate* $I = \iint_D 30x^2y\,\mathrm{d}A$, *where* D *is the region bounded by the lines* $x = 2, y = x$ *and the hyperbola* $xy = 1$ *in two different ways (i.e. considering* D *as a type I and then as a type II region).*

EXAMPLE 6. *Find the volume of the solid bounded by the cylinder* $x^2 + y^2 = 1$ *and the planes* $x = 0, y = z, z = 0$ *in the first octant.*

EXAMPLE 7. *Evaluate the following iterated integral by reversing the order of integration:*

$$I = \int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, \mathrm{d}y \mathrm{d}x$$

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