15.3: Double Integrals in Polar Coordinates

The **polar coordinate system** consists of:

- the **pole** (or origin) labeled O;
- the **polar axis** which is a ray starting at O (usually drawn horizontally to the right);

The **polar coordinates** (r, θ) of a point P:

- θ is the angle between the polar axis and the line OP (the angle is positive if measured in counterclockwise direction from the polar axis);
- r is the distance from O to P.

EXAMPLE 1. *Plot the points whose polar coordinates are given:*

(a) $(1, \pi/3)$ **(b)** $(5, -\pi/2)$.

The connection between polar and Cartesian coordinates:

$\cos\theta =$ $\sin\theta =$

$x =$ $y =$

$r^2 =$ $\tan\theta =$

REMARK 2. In converting from the Cartesian to polar coordinates we must choose θ so that the point (r, θ) lies in the correct quadrant.

EXAMPLE 3. *What curve is represented by the following polar equation*

(a) $r = 12$ *(b)* $\theta = \frac{\pi}{3}$

EXAMPLE 4. *Sketch the region in the Cartesian plane consisting of points whose polar coordinates satisfy the following conditions:* $1 \le r \le 2, \quad \pi/4 \le \theta \le \pi$.

EXAMPLE 5. *Find a polar equation for the curve represented by the given Cartesian equation:*

(a) $x^2 + y^2 = 2by$

(b) $(x - a)^2 + y^2 = a^2$

Using polar coordinates to evaluate double integrals

EXAMPLE 6. *Evaluate*

$$I = \iint_D \arctan \frac{y}{x}\, \mathrm{d}A$$

where $D = \left\{(x, y)|\ 1 \leq x^2 + y^2 \leq 4, x \leq y \leq \sqrt{3}x, x \geq 0\right\}$.

THEOREM 7. *Change to polar coordinates in a double integral: Let f be a continuous on the region D. Denote by D^* the region representing D in the polar coordinates (r, θ). Then*

$$\iint_D f(x, y)\, \mathrm{d}A = \iint_{D^*} f(r\cos\theta, r\sin\theta)\ r\, \mathrm{d}r\mathrm{d}\theta.$$

REMARK 8. Be careful not to forget the additional factor r on the right side of the formula.

Solution of Example 6:

Evaluate $I = \iint_D \arctan \frac{y}{x} \, \mathrm{d}A$, where $D = \left\{(x, y) |\ 1 \leq x^2 + y^2 \leq 4, x \leq y \leq \sqrt{3}x, x \geq 0\right\}$.

EXAMPLE 9. *Find the volume of the solid that lies under the paraboloid* $z = x^2 + y^2$, *above the xy-plane and inside the cylinder* $x^2 + y^2 = 2x$.

EXAMPLE 10. *Find the area of the region inside the circle $r = 4\sin\theta$ and outside the circle $r = 2$.*