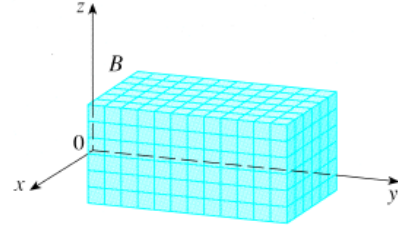


15.6: Triple Integrals

Mass problem: Given a solid object, that occupies the region B in \mathbb{R}^3 , with density $\rho(x, y, z)$. Find the mass of the object.

Solution: Let B be a rectangular box:

$$B = \{(x, y, z) | a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$$



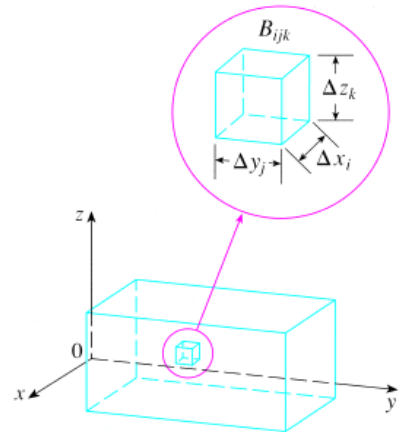
Partition in sub-boxes:

$$m_{ijk} = \rho(x_i^*, y_j^*, z_k^*) \Delta V_{ijk}$$

$$\|P\| = \max \sqrt{\Delta x_i^2 + \Delta y_j^2 + \Delta z_k^2}$$

$$m = \lim_{\|P\| \rightarrow 0} \sum_i \sum_j \sum_k \rho(x_i^*, y_j^*, z_k^*) \Delta V_{ijk}$$

$$m = \iiint_B \rho(x, y, z) dV$$



1

FUBINI'S THEOREM: If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$ then

$$\iiint_B f(x, y, z) dV = \int_r^s \int_c^d \int_a^b f(x, y, z) dx dy dz$$

and there are 5 other possible orders in which we can integrate.

EXAMPLE 1. Let $B = [0, 1] \times [-1, 3] \times [0, 3]$. Evaluate

$$I = \iiint_B xye^{yz} dV$$

FACT: The volume of the solid E is given by the integral,

$$V = \iiint_E dV.$$

FACT: The mass of the solid E with variable density $\rho(x, y, z)$ is given by the integral,

$$m = \iiint_E \rho(x, y, z) dV.$$

EXAMPLE 2. Find the mass of the solid bounded by $x = y^2 + z^2$ and the plane $x = 4$ if the density function is $\rho(x, y, z) = \sqrt{y^2 + z^2}$.

EXAMPLE 3. Use a triple integral to find the volume of the solid bounded by the surfaces $z = x^2 + y^2$ and $z = 5 - 4x^2 - 4y^2$.

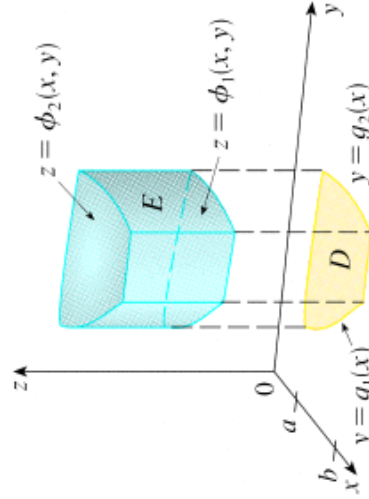
EXAMPLE 4. Use a triple integral to find the volume of the solid bounded by the elliptic cylinder $4x^2 + z^2 = 4$ and the planes $y = 0$ and $y = z + 2$.

Table 1: Triple integrals over a general bounded region E

A solid region of **TYPE I**:

$E = \{(x, y, z) | (x, y) \in D, \phi_1(x, y) \leq z \leq \phi_2(x, y)\}$
 where D is the projection of E onto the xy -plane.

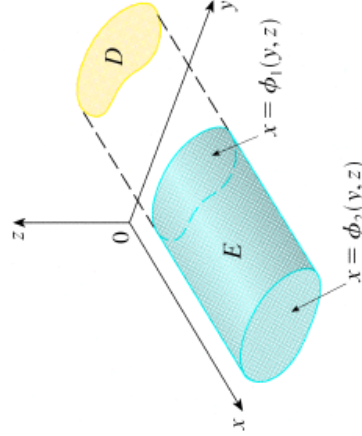
A type 1 solid region



$$\iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{\phi_1(x, y)}^{\phi_2(x, y)} f(x, y, z) \, dz \right] \, dA$$

A solid region of **TYPE II**:

$E = \{(x, y, z) | (y, z) \in D, \phi_1(y, z) \leq x \leq \phi_2(y, z)\}$
 where D is the projection of E onto the yz -plane.

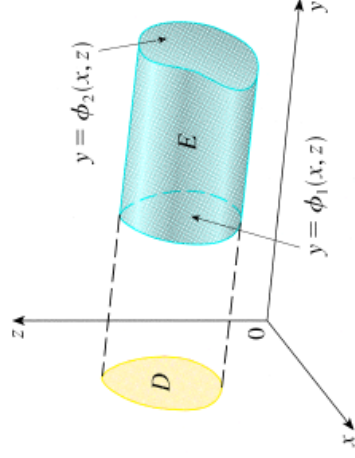


A type 2 region

$$\iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{\phi_1(y, z)}^{\phi_2(y, z)} f(x, y, z) \, dx \right] \, dA$$

A solid region of **TYPE III**:

$E = \{(x, y, z) | (x, z) \in D, \phi_1(x, z) \leq y \leq \phi_2(x, z)\}$
 where D is the projection of E onto the xz -plane.



A type 3 region

$$\iiint_E f(x, y, z) \, dV =$$

When we set up a triple integral it is wise to draw **two** diagrams: one of the solid region E and one of its projection on the corresponding coordinate plane.