15.7: Triple integrals in cylindrical coordinates

- **Cylindrical coordinates:**

$P(x, y, z) \in \mathbb{R}^3$
In the cylindrical coordinates P is represented by the ordered triple (r, θ, z), where r, θ are the polar coordinates of P_{xy} and z is the directed distance from the xy-plane to P:

$$x = \qquad\qquad y = \qquad\qquad z = \qquad ,$$

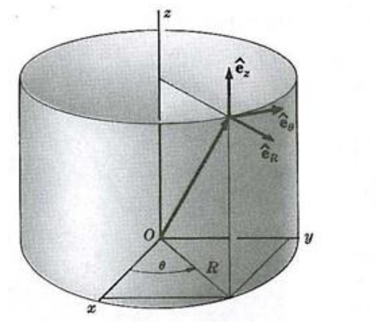
where

$$r^2 = \qquad\qquad \tan\theta = \qquad\qquad z = z.$$

REMARK 1. The cylindrical coordinates

$$\begin{aligned} &x = r\cos\theta \\ &y = r\sin\theta \\ &z = z \\ &r \geq 0, \quad 0 \leq \theta \leq 2\pi \end{aligned}$$

are useful in problems that involve *symmetry about the z-axis.*



(a) Cylindrical

https://i.stack.imgur.com/FgSBF.jpg

EXAMPLE 2. *Find an equation in cylindrical coordinates for the cone*

$$z = \sqrt{x^2 + y^2}$$

THEOREM 3. *Let $f(x, y, z)$ be a continuous function over a solid $E \subset \mathbb{R}^3$. Let E^* be its image in cylindrical coordinates. Then*

$$\iiint_E f(x, y, z)\,\mathrm{d}V = \iiint_{E^*} f(r\cos\theta, r\sin\theta, z)\,\mathrm{d}V^*,$$

where

$$\mathrm{d}V^* = r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta.$$

EXAMPLE 4. *The density at any point of the solid E,*

$$E = \left\{(x, y, z) : x^2 + y^2 \leq 9, -1 \leq z \leq 4\right\},$$

equals to its distance from the axis of E. Find the mass of E.

REMARK 5. If E is a solid region of type I, i.e.

$$E = \{(x, y, z) | (x, y) \in D, \phi_1(x, y) \leq z \leq \phi_2(x, y)\},$$

where D is the projection of E onto the xy-plane then, as we know,

$$\iiint_E f(x, y, z)\, \mathrm{d}V = \iint_D \left[\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x, y, z)\, \mathrm{d}z\right] \mathrm{d}A.$$

Passing to cylindrical coordinates here we actually have to replace D by its image D^* in polar coordinates and $\mathrm{d}z\, \mathrm{d}A$ by $r\, \mathrm{d}z\, \mathrm{d}r\, \mathrm{d}\theta$.

EXAMPLE 6. *Find the volume of the solid E bounded by the surfaces*

$$y = x, \quad y = -x, \quad x^2 + y^2 = 5z, \quad z = 7$$

so that $y \geq 0$.