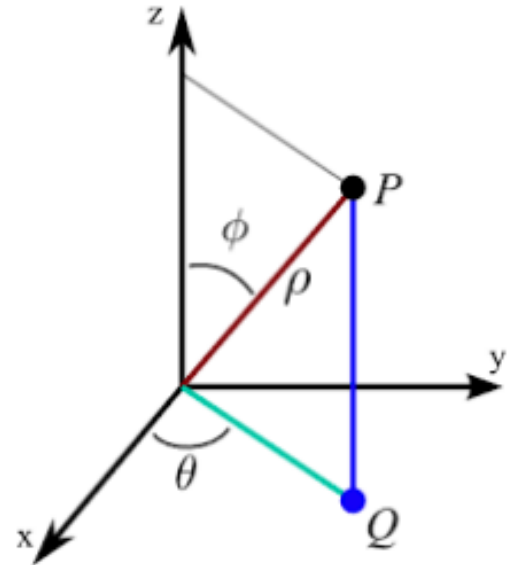


15.8: Triple integrals in spherical coordinates

- **Spherical coordinates** of P is the ordered triple (ρ, θ, ϕ) where $|OP| = \rho$, $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$.



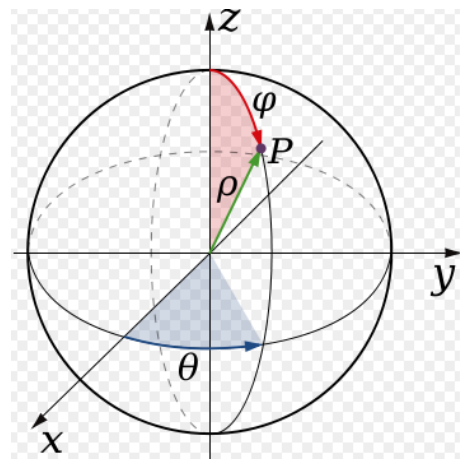
The spherical coordinates

$$\begin{aligned} x &= \rho \sin \phi \cos \theta \\ y &= \rho \sin \phi \sin \theta \\ z &= \rho \cos \phi \\ \rho &\geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi \end{aligned}$$

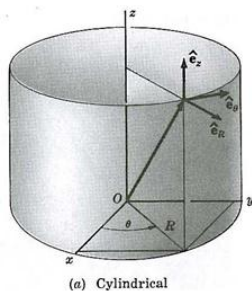
are especially useful in problems where there is *symmetry about the origin*.

Note that

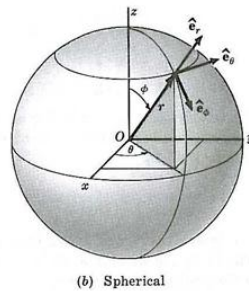
$$x^2 + y^2 + z^2 =$$



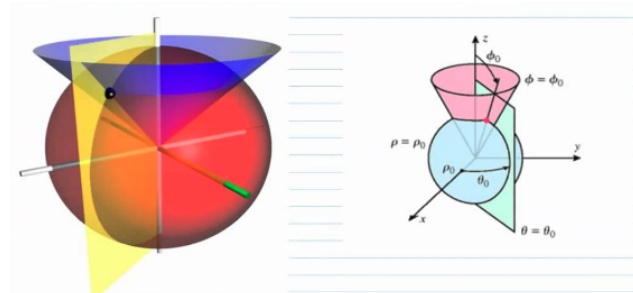
<https://commons.wikimedia.org/wiki/>



(a) Cylindrical



(b) Spherical



<https://i.stack.imgur.com/FgSBF.jpg>

<https://www.youtube.com/watch?v=Q-RUZIboBeE>

EXAMPLE 1. Find equation in spherical coordinates for the following surfaces.

(a) $x^2 + y^2 + z^2 = 16$

(b) $z = \sqrt{x^2 + y^2}$

(c) $z = \sqrt{3x^2 + 3y^2}$

(d) $x = y$

•Triple integrals in spherical coordinates

$$\begin{aligned}x &= \rho \sin \phi \cos \theta \\y &= \rho \sin \phi \sin \theta \\z &= \rho \cos \phi \\ \rho &\geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi\end{aligned}$$

$$\begin{array}{ccccc}(x, y, z) & \rightarrow & (r, \theta, z) & \rightarrow & (\rho, \theta, \phi) \\x = r \cos \theta & & & & z = \rho \cos \phi \\y = r \sin \theta & & & & r = \rho \sin \phi \\z = z & & & & \theta = \theta\end{array}$$

$$dV = dx dy dz =$$

THEOREM 2. Let $f(x, y, z)$ be a continuous function over a solid $E \subset \mathbb{R}^3$. Let E^* be its image in spherical coordinates. Then

$$\iiint_E f(x, y, z) dV = \iiint_{E^*} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\theta d\phi.$$

EXAMPLE 3. Evaluate $I = \iiint_E e^{\sqrt{(x^2+y^2+z^2)^3}} dV$ where $E = \{(x, y, z) : 9 \leq x^2 + y^2 + z^2 \leq 16\}$.

EXAMPLE 4. Write the integral $\iiint_E f(x, y, z) \, dV$ in spherical coordinates where

(a) $E = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1, y \geq 0, z \geq 0\}$.

(b) E is the ice cream cone-shaped solid, which is cut from the sphere of radius 5 by the cone $\phi = \pi/6$.

EXAMPLE 5. Evaluate the integral by changing to spherical coordinates:

$$I = \int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2 + y^2 + z^2) \, dz \, dx \, dy.$$