15.9: Change Of Variables In Double Integral

Examples of a change of variables:

- substitution rule

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_\alpha^\beta f(u)\,\mathrm{d}u.$$

- conversion to polar coordinates:

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D^*} f(r\cos\theta, r\sin\theta)\ r\,\mathrm{d}r\mathrm{d}\theta.$$

- conversion to cylindrical coordinates:

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iiint_{E^*} f(r\cos\theta, r\sin\theta, z)r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta$$

- conversion to spherical coordinates:

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iiint_{E^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

We call the equations that define the change of variables a **transformation:**

$$x = x(u,v), \quad y = y(u,v).$$

EXAMPLE 1. *Determine the new region that we get by applying the transformation* $x = 3u, y = u/2$ *to the region* $D = \left\{(x,y)\middle|\ \frac{x^2}{36} + y^2 = 1\right\}$.

DEFINITION 2. *The* **Jacobian** *of the transformation* $x = x(u,v), y = y(u,v)$ *is*

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

EXAMPLE 3. *Compute the Jacobian of the transformation* $x = r\cos\theta,\ y = r\sin\theta$.

Change of variables for a double integral:

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v.$$

Explanation

EXAMPLE 4. *Evaluate*

$$\iint_D e^{\frac{y-x}{y+x}}\,\mathrm{d}A$$

where D *is triangle with vertices* $(0,0)$, $(2,0)$, $(0,2)$.

EXAMPLE 5. *Find mass of a lamina that occupies the region*

$$D = \left\{(x, y) | 16 \leq x^2 + y^2 \leq 25, \quad 1 \leq x^2 - y^2 \leq 9, \quad x \geq 0, y \geq 0\right\}$$

with density $\rho(x, y) = 8xy$.