16.1: Vector Fields

A vector function

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

is an example of a function whose domain is a set of real numbers and whose range is a set of vectors in $\mathbb{R}^3$:

$$\mathbf{r}(t) : \mathbb{R} \to \mathbb{R}^3.$$

A vector fields on $\mathbb{R}^2$ (or $\mathbb{R}^3$) is an assignment so that to every point of $\mathbb{R}^2$ (or of $\mathbb{R}^3$) one assigns a vector in $\mathbb{R}^2$ (or $\mathbb{R}^3$). Usually one attaches the starting point of this vector to the point to which this vector is assigned.

Vector field on $\mathbb{R}^2$.

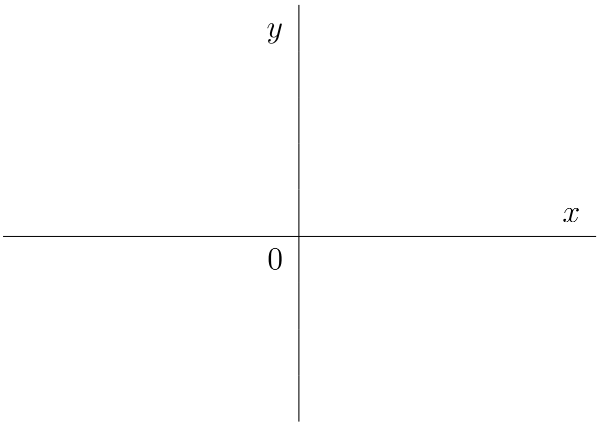
$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \langle P(x, y), Q(x, y)\rangle$$

Vector field on $\mathbb{R}^3$:

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k} = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$$

A vector field in the plane (for instance), can be visualized as a collection of arrows with a given magnitude and direction, each attached to a point in the plane.

EXAMPLE 1. *Describe the vector field* $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ *by sketching.*



Vector fields are often used to model, for example, the speed and direction of a moving fluid throughout space, or the strength and direction of some force, such as the magnetic or gravitational force, as it changes from one point to another point.

EXAMPLE 2. **Gravitational Field:**

By Newton's Law of Gravitation the magnitude of the gravitational force between two objects with masses m *and* M *is The gravitational force acting on the object at* (x, y, z) *is*

$$|\mathbf{F}| = G\frac{mM}{r^2},$$

where $r = \sqrt{x^2 + y^2 + z^2}$ *is the distance between the objects and* G *is the gravitational constant.*

Function $u = f(x, y, z)$ is also called a **scalar field**. Its gradient is also called **gradient vector field**:

$$\mathbf{F}(x, y, z) = \nabla f(x, y, z) =$$

EXAMPLE 3. *Find the gradient vector field of* $f(x, y, z) = xyz$.

DEFINITION 4. *A vector field* $\mathbf{F}$ *is called a* **conservative vector field** *if it is the gradient of some scalar function* f *s.t* $\mathbf{F} = \nabla f$. *In this situation* f *is called a* **potential function** *for* $\mathbf{F}$.

For instance, the vector field $\mathbf{F}(x, y) = \langle x, y\rangle$ is a conservative vector field with a potential function $f(x, y) = xy$ because

REMARK 5. Not all vector fields are conservative, but such fields do arise frequently in Physics.

EXAMPLE 6. *(see Example 2)Let*

$$f(x, y, z) = \frac{GmM}{r},$$

where $r = \sqrt{x^2 + y^2 + z^2}$. *Find its gradient and and answer the questions:*

(a) *Is the gravitational field conservative?*

(b) *What is a potential function of the gravitational field?*