

16.5: Curl and Divergence

Introduce the vector differential operator ∇ as

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}.$$

If $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ is a vector field on \mathbb{R}^3 and the partial derivatives of P, Q, R all exist, then the **curl** of \mathbf{F} is the *vector field* on \mathbb{R}^3 defined by

$$\text{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

EXAMPLE 1. Find the curl of the vector field

$$\mathbf{F}(x, y, z) = \langle xy, x^2, yz \rangle.$$

Question What is the curl of a two-dimensional vector field

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} ?$$

Answer:

CONCLUSION: Green's Theorem in vector form:

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} =$$

THEOREM 2. If a function $f(x, y, z)$ has continuous partial derivatives of second order then

$$\operatorname{curl}(\nabla f) = \mathbf{0}.$$

Proof.

COROLLARY 3. If \mathbf{F} is conservative, then $\operatorname{curl}\mathbf{F} = \mathbf{0}$.

The proof of the Theorem below requires Stokes' Theorem (Section 14.8).

THEOREM 4. If \mathbf{F} is a vector field defined on \mathbb{R}^3 whose component functions have continuous partial derivatives and $\operatorname{curl}\mathbf{F} = \mathbf{0}$, then \mathbf{F} is a conservative vector field.

EXAMPLE 5. Let $\mathbf{F}(x, y, z) = \langle x^9, y^9, z^9 \rangle$.

(a) Show that \mathbf{F} is conservative.

(b) Find a function f s.t. $\nabla f = \mathbf{F}$.

(c) Evaluate $\int_{(1,0,1)}^{(-1,-1,-1)} \mathbf{F} \cdot d\mathbf{r}$.

If $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ is a vector field on \mathbb{R}^3 and the partial derivatives P_x, Q_y, R_z exist, then the **divergence of \mathbf{F}** is the *scalar field* on defined by

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} =$$

EXAMPLE 6. Find the divergence of the vector field

$$\mathbf{F}(x, y, z) = \langle \sin(xyz), x^2, yz \rangle.$$

THEOREM 7. If the components of a vector field $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ has continuous partial derivatives of second order then

$$\operatorname{div} \operatorname{curl} \mathbf{F} = 0.$$

Proof.

EXAMPLE 8. Is there a vector field \mathbf{G} on \mathbb{R}^3 s.t. $\operatorname{curl} G = \langle yz, xyz, zy \rangle$?