

16.6: Parametric surfaces and their areas

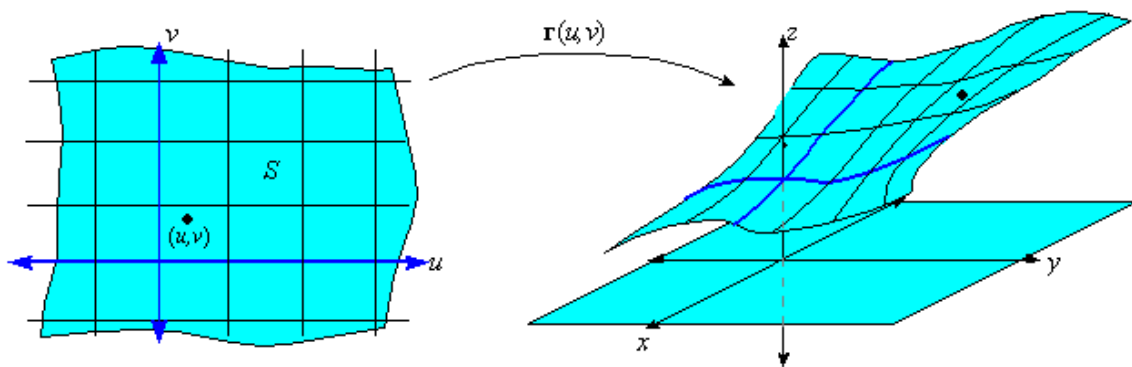
Consider a continuous vector valued function of two variables

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}, \quad (u, v) \in D.$$

Parametric surface:

$$S : x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad (u, v) \in D.$$

In other words, the surface S is traces out by the position vector $\mathbf{r}(u, v)$ as (u, v) moves throughout the region D .



EXAMPLE 1. Determine the surface given by the parametric representation

$$\mathbf{r}(u, v) = \langle u, u \cos v, u \sin v \rangle, \quad 1 \leq u \leq 5, \quad 0 \leq v \leq 2\pi$$

EXAMPLE 2. Give parametric or vector representations for each of the following surfaces:

(a) the cylinder: $x^2 + y^2 = 9$, $1 \leq z \leq 5$.

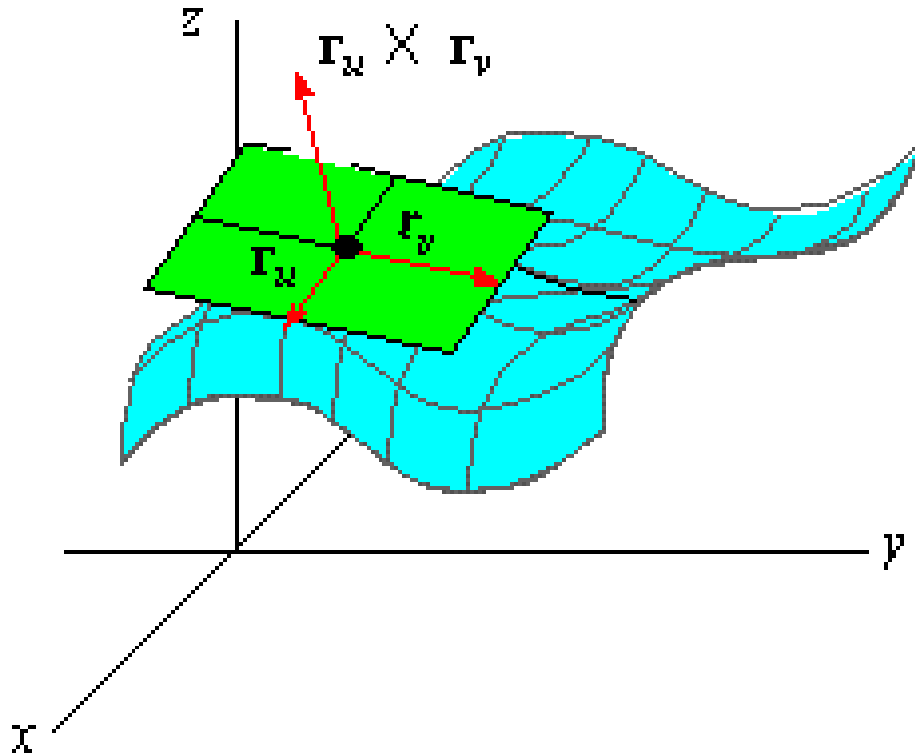
(b) the upper half-sphere: $z = \sqrt{100 - x^2 - y^2}$.

CONCLUSION: To parametrize surfaces in majority of the exercises that will be given you may use polar, cylindrical or spherical coordinates, or

- $z = f(x, y) \longrightarrow \mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$
- $y = f(x, z) \longrightarrow \mathbf{r}(x, z) = x\mathbf{i} + f(x, z)\mathbf{j} + z\mathbf{k}$
- $x = f(y, z) \longrightarrow \mathbf{r}(y, z) = f(y, z)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

- **Tangent planes:**

PROBLEM: Find a normal vector to the tangent plane to a parametric surface S given by a vector function $\mathbf{r}(u, v)$ at a point P_0 with position vector $\mathbf{r}(u_0, v_0)$, i.e. $P_0(x(u_0, v_0), y(u_0, v_0), z(u_0, v_0))$



The normal vector

$$\mathbf{N} = \mathbf{N}(u, v) =$$

If a normal vector is not $\mathbf{0}$ then the surface S is called **smooth** (it has no "corner").

Special Case: a surface S given by a graph $z = f(x, y)$. Then one can choose the following parametrization of S :

$$\mathbf{r}(x, y) =$$

and then the normal vector is

$$\mathbf{N} =$$

EXAMPLE 3. Find the tangent plane to the surface with parametric equations $x = uv + 1, y = ue^v, z = ve^u$ at the point $(1, 0, 0)$.

• **Surface Area:**

Consider a smooth surface S given by

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}, \quad (u, v) \in D,$$

then

$$dS = |N(u, v)|du dv =$$

and the **surface area**

$$A(S) = \iint_S dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA.$$

REMARK 4. *Special Case:* a surface S given by a graph $z = f(x, y)$ we have

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$$

and

$$dS = |\mathbf{N}(x, y)| dA =$$

EXAMPLE 5. *Find the surface area of the surface*

$$S: \quad x = uv, \quad y = u + v, \quad z = u - v, \quad u^2 + v^2 \leq 1.$$

EXAMPLE 6. Find the surface area of the part paraboloid $z = x^2 + y^2$ between two planes: $z = 0$ and $z = 4$.