Projective and affine equivalence of sub-Riemannian metrics, part 2: separation on the level of nilpotent approximation and Jacobi curves, generic projective rigidity and Weyl type theorems.

Igor Zelenko

Texas A&M University

based on the joint work with Frederic Jean (ENSTA, Paris) and Sofya Maslovskaya (INRIA, Sophia Antipolis)

Seminario de Singularidades UFC Online, July , 2020

Sub-Riemannian metrics

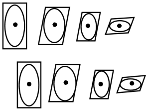
A rank ℓ distribution $D = \{D(q)\}_{q \in M}$ on a manifold M is a rank ℓ subbundle of the tangent bundle TM (a smooth field of ℓ-dimensional subspaces $D(q)$ of the tangent spaces T_qM).

D is called bracket-generating distribution if at any point iterated Lie brackets of vector fields tangent to D generate the whole tangent space.

Rashevsky-Chow Any two points of M can be connected by a curve tangent to a distribution.

A sub-Riemannian metric g is given on the distribution D, if an inner product g_q is chosen on each subspaces $D(q)$ smoothly in q.

Riemannian case: $D = TM$

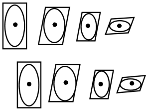


Sub-Riemannian metrics

A rank ℓ distribution $D = \{D(q)\}_{q\in M}$ on a manifold M is a rank ℓ subbundle of the tangent bundle TM (a smooth field of ℓ-dimensional subspaces $D(q)$ of the tangent spaces T_qM).
D is called bracket-generating distribution if at any point iterated Lie brackets of vector fields tangent to D generate the whole tangent space.
Rashevsky-Chow Any two points of M can be connected by a curve tangent to a distribution.

A sub-Riemannian metric g is given on the distribution D, if an inner product g_q is chosen on each subspaces $D(q)$ smoothly in q.

Riemannian case: $D = TM$

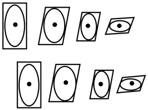


Sub-Riemannian metrics

A rank ℓ distribution $D = \{D(q)\}_{q \in M}$ on a manifold M is a rank ℓ subbundle of the tangent bundle TM (a smooth field of ℓ-dimensional subspaces $D(q)$ of the tangent spaces T_qM).
D is called bracket-generating distribution if at any point iterated Lie brackets of vector fields tangent to D generate the whole tangent space.
Rashevsky-Chow Any two points of M can be connected by a curve tangent to a distribution.

A sub-Riemannian metric g is given on the distribution D, if an inner product g_q is chosen on each subspaces $D(q)$ smoothly in q.

Riemannian case: $D = TM$

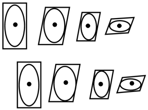


Sub-Riemannian metrics

A rank ℓ distribution $D = \{D(q)\}_{q \in M}$ on a manifold M is a rank ℓ subbundle of the tangent bundle TM (a smooth field of ℓ-dimensional subspaces $D(q)$ of the tangent spaces T_qM).
D is called bracket-generating distribution if at any point iterated Lie brackets of vector fields tangent to D generate the whole tangent space.
Rashevsky-Chow Any two points of M can be connected by a curve tangent to a distribution.

A sub-Riemannian metric g is given on the distribution D, if an inner product g_q is chosen on each subspaces $D(q)$ smoothly in q.

Riemannian case: $D = TM$

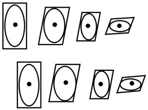


Sub-Riemannian metrics

A rank ℓ distribution $D = \{D(q)\}_{q \in M}$ on a manifold M is a rank ℓ subbundle of the tangent bundle TM (a smooth field of ℓ-dimensional subspaces $D(q)$ of the tangent spaces T_qM).
D is called bracket-generating distribution if at any point iterated Lie brackets of vector fields tangent to D generate the whole tangent space.
Rashevsky-Chow Any two points of M can be connected by a curve tangent to a distribution.

A sub-Riemannian metric g is given on the distribution D, if an inner product g_q is chosen on each subspaces $D(q)$ smoothly in q.

Riemannian case: $D = TM$



Sub-Riemannian geodesics

Given a sub-Riemannian (sR) metric g, for any curve γ tangent to the distribution one can define the sub-Riemannian length by $\int g\big(\dot\gamma(t),\dot\gamma(t)\big)^{\frac{1}{2}}dt$.

Sub-Riemannian geodesics are the candidates for length-minimizers (via the Pontryagin Maximum Principle in Optimal Control).

Two types of geodesics: normal and abnormal geodesics (the latter depend on the distribution D but not on the metric as unparametrized curves; no such geodesics in Riemannian case).

Sub-Riemannian geodesics

Given a sub-Riemannian (sR) metric g, for any curve γ tangent to the distribution one can define the sub-Riemannian length by $\int g\big(\dot\gamma(t),\dot\gamma(t)\big)^{\frac{1}{2}}dt$.

Sub-Riemannian geodesics are the candidates for length-minimizers (via the Pontryagin Maximum Principle in Optimal Control).

Two types of geodesics: normal and abnormal geodesics (the latter depend on the distribution D but not on the metric as unparametrized curves; no such geodesics in Riemannian case).

Sub-Riemannian geodesics

Given a sub-Riemannian (sR) metric g, for any curve γ tangent to the distribution one can define the sub-Riemannian length by $\int g\big(\dot\gamma(t),\dot\gamma(t)\big)^{\frac{1}{2}}dt$.

Sub-Riemannian geodesics are the candidates for length-minimizers (via the Pontryagin Maximum Principle in Optimal Control).

Two types of geodesics: normal and abnormal geodesics (the latter depend on the distribution D but not on the metric as unparametrized curves; no such geodesics in Riemannian case).

Sub-Riemannian geodesics

Given a sub-Riemannian (sR) metric g, for any curve γ tangent to the distribution one can define the sub-Riemannian length by $\int g\big(\dot{\gamma}(t), \dot{\gamma}(t)\big)^{\frac{1}{2}} dt$.

Sub-Riemannian geodesics are the candidates for length-minimizers (via the Pontryagin Maximum Principle in Optimal Control).

Two types of geodesics: normal and abnormal geodesics (the latter depend on the distribution D but not on the metric as unparametrized curves; no such geodesics in Riemannian case).

Sub-Riemannian geodesics

Given a sub-Riemannian (sR) metric g, for any curve γ tangent to the distribution one can define the sub-Riemannian length by $\int g\big(\dot\gamma(t), \dot\gamma(t)\big)^{\frac{1}{2}} dt$.

Sub-Riemannian geodesics are the candidates for length-minimizers (via the Pontryagin Maximum Principle in Optimal Control).

Two types of geodesics: normal and abnormal geodesics (the latter depend on the distribution D but not on the metric as unparametrized curves; no such geodesics in Riemannian case).

Sub-Riemannian geodesics: in more details

sR Hamiltonian is the function $h_g : T^*M \to \mathbb{R}$ defined by

$$h_g(q,p) = \frac{1}{2}\max\left\{\langle p, v\rangle^2 : v \in D(q),\ g(q)(v,v) = 1\right\}, \quad q \in M,\ p \in T_q^*M$$

(quadratic form on the fiber T_q^*M)

- Normal extremals are trajectories $\lambda(\cdot)$ of the Hamiltonian vector field on a **nonzero level set** of h_g,

$$\lambda(t) = e^{t\vec{h}_g}\lambda \qquad \text{for some } \lambda \in T^*M$$

- Abnormal extremal: Lipshitzian curves in the **zero level set** of h_g ($= D^\perp$) such that their tangent lines at almost every point belong to the $\ker \sigma|_{D^\perp}$, where σ is the canononical symplectic form on T^*M.

Normal/abnormal geodesic are projections of normal/abnormal extremals (from T^*M to M).

Sub-Riemannian geodesics: in more details

sR Hamiltonian is the function $h_g : T^*M \to \mathbb{R}$ defined by

$$h_g(q,p) = \frac{1}{2}\max\left\{\langle p, v\rangle^2 : v \in D(q),\ g(q)(v,v) = 1\right\}, \quad q \in M,\ p \in T_q^*M$$

(quadratic form on the fiber T_q^*M)

- Normal extremals are trajectories $\lambda(\cdot)$ of the Hamiltonian vector field on a **nonzero level set** of h_g,

$$\lambda(t) = e^{t\vec{h}_g}\lambda \qquad \text{for some } \lambda \in T^*M$$

- Abnormal extremal: Lipshitzian curves in the **zero level set** of h_g $(= D^{\perp})$ such that their tangent lines at almost every point belong to the $\ker \sigma|_{D^{\perp}}$, where σ is the canononical symplectic form on T^*M.

Normal/abnormal geodesic are projections of normal/abnormal extremals (from T^*M to M).

Sub-Riemannian geodesics: in more details

sR Hamiltonian is the function $h_g : T^*M \to \mathbb{R}$ defined by

$$h_g(q,p) = \frac{1}{2}\max\left\{\langle p, v\rangle^2 : v \in D(q),\ g(q)(v,v) = 1\right\}, \quad q \in M,\ p \in T_q^*M$$

(quadratic form on the fiber T_q^*M)

- Normal extremals are trajectories $\lambda(\cdot)$ of the Hamiltonian vector field on a **nonzero level set** of h_g,

  $$\lambda(t) = e^{t\vec{h}_g}\lambda \qquad \text{for some } \lambda \in T^*M$$

- Abnormal extremal: Lipshitzian curves in the **zero level set** of h_g ($= D^\perp$) such that their tangent lines at almost every point belong to the $\ker \sigma|_{D^\perp}$, where σ is the canononical symplectic form on T^*M.

Normal/abnormal geodesic are projections of normal/abnormal extremals (from T^*M to M).

Sub-Riemannian geodesics: in more details

sR Hamiltonian is the function $h_g : T^*M \to \mathbb{R}$ defined by

$$h_g(q,p) = \frac{1}{2}\max\left\{\langle p, v\rangle^2 : v \in D(q),\ g(q)(v,v) = 1\right\}, \quad q \in M,\ p \in T_q^*M$$

(quadratic form on the fiber T_q^*M)

- Normal extremals are trajectories $\lambda(\cdot)$ of the Hamiltonian vector field on a **nonzero level set** of h_g,

$$\lambda(t) = e^{t\vec{h}_g}\lambda \qquad \text{for some } \lambda \in T^*M$$

- Abnormal extremal: Lipshitzian curves in the **zero level set** of h_g $(= D^\perp)$ such that their tangent lines at almost every point belong to the $\ker \sigma|_{D^\perp}$, where σ is the canononical symplectic form on T^*M.

Normal/abnormal geodesic are projections of normal/abnormal extremals (from T^*M to M).

Projective/affine equivalence and existence of orbital diffeomorphism

Definition

Two sub-Riemannian metrics g_1 and g_2 on a distribution D are called projectively/affinely equivalent if they have the same normal geodesics, up to a reparametrization/an affine parametrization.

Let g and $\tilde{g}$ be two metrics on D.

Orbital diffeomorphism between $\vec{h}_g$ and $\vec{h}_{\tilde{g}}$ = local fiber-preserving diffeomorphism $\Phi : T^*M \to T^*M$ such that $\Phi\big(e^{t\vec{h}_g}\lambda\big) = e^{s\vec{h}_{\tilde{g}}}\big(\Phi(\lambda)\big)$, i.e.

$$\Phi_*\vec{h}_g = a\vec{h}_{\tilde{g}}, \qquad a \in C^\infty(T^*M)$$

Proposition

If g, $\tilde{g}$ projectively equivalent, then $\vec{h}_g$, $\vec{h}_{\tilde{g}}$ orbitally diffeomorphic near generic point of T^*M

Projective/affine equivalence and existence of orbital diffeomorphism

Definition

Two sub-Riemannian metrics g_1 and g_2 on a distribution D are called projectively/affinely equivalent if they have the same normal geodesics, up to a reparametrization/an affine parametrization.

Let g and $\tilde{g}$ be two metrics on D.

Orbital diffeomorphism between $\vec{h}_g$ and $\vec{h}_{\tilde{g}}$ = local fiber-preserving diffeomorphism $\Phi : T^*M \to T^*M$ such that $\Phi\big(e^{t\vec{h}_g}\lambda\big) = e^{s\vec{h}_{\tilde{g}}}\big(\Phi(\lambda)\big)$, i.e.

$$\Phi_*\vec{h}_g = a\vec{h}_{\tilde{g}}, \qquad a \in C^\infty(T^*M)$$

Proposition

If g, $\tilde{g}$ projectively equivalent, then $\vec{h}_g$, $\vec{h}_{\tilde{g}}$ orbitally diffeomorphic near generic point of T^*M

Projective/affine equivalence and existence of orbital diffeomorphism

Definition

Two sub-Riemannian metrics g_1 and g_2 on a distribution D are called projectively/affinely equivalent if they have the same normal geodesics, up to a reparametrization/an affine parametrization.

Let g and $\tilde{g}$ be two metrics on D.

Orbital diffeomorphism between $\vec{h}_g$ and $\vec{h}_{\tilde{g}}$ = local fiber-preserving diffeomorphism $\Phi : T^*M \to T^*M$ such that $\Phi\left(e^{t\vec{h}_g}\lambda\right) = e^{s\vec{h}_{\tilde{g}}}\left(\Phi(\lambda)\right)$, i.e.

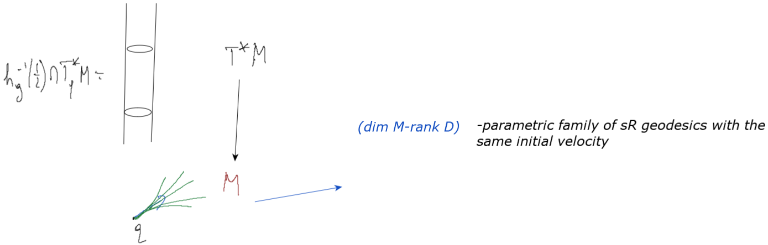
$$\Phi_*\vec{h}_g = a\vec{h}_{\tilde{g}}, \qquad a \in C^\infty(T^*M)$$

Proposition

If g, $\tilde{g}$ projectively equivalent, then $\vec{h}_g$, $\vec{h}_{\tilde{g}}$ orbitally diffeomorphic near generic point of T^*M

The idea of the proof: recovery of an extremal from a sufficiently high jet of the geodesic

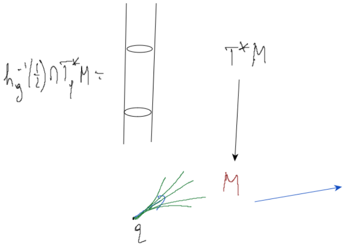
All sR normal geodesics γ starting at a given point $q \in M$ with $||\dot{\gamma}(0)|| = 1$ are “parametrized” by points of the cylinder $h_g^{-1}(1/2) \cap T_q^* M$:



In different directions of D sR geodesics may be distinguished by jets of different order. For an even contact distribution there is a special (characteristic) direction C s. t. all geodesics γ with the same initial $\dot{\gamma}(0)$ not in this direction are distinguished by the 2nd jet, but the 2nd jet of all geodesics with $\dot{\gamma}(0)$ in the direction of C coincide.

The idea of the proof: recovery of an extremal from a sufficiently high jet of the geodesic

All sR normal geodesics γ starting at a given point $q \in M$ with $||\dot\gamma(0)|| = 1$ are “parametrized” by points of the cylinder $h_g^{-1}(1/2) \cap T_q^* M$:

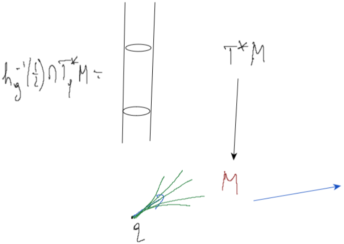


(dim M-rank D) *-parametric family of sR geodesics with the same initial velocity*

In different directions of D sR geodesics may be distinguished by jets of different order. For an even contact distribution there is a special (characteristic) direction C s. t. all geodesics γ with the same initial $\dot\gamma(0)$ not in this direction are distinguished by the 2nd jet, but the 2nd jet of all geodesics with $\dot\gamma(0)$ in the direction of C coincide.

The idea of the proof: recovery of an extremal from a sufficiently high jet of the geodesic

All sR normal geodesics γ starting at a given point $q \in M$ with $||\dot{\gamma}(0)|| = 1$ are "parametrized" by points of the cylinder $h_g^{-1}(1/2) \cap T_q^* M$:

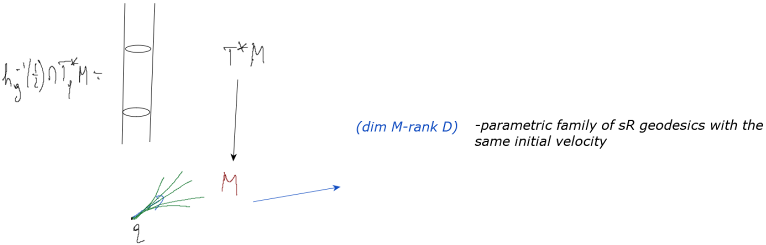


(dim M-rank D) -parametric family of sR geodesics with the same initial velocity

In different directions of D sR geodesics may be distinguished by jets of different order. For an even contact distribution there is a special (characteristic) direction C s. t. all geodesics γ with the same initial $\dot{\gamma}(0)$ not in this direction are distinguished by the 2nd jet, but the 2nd jet of all geodesics with $\dot{\gamma}(0)$ in the direction of C coincide.

The idea of the proof: recovery of an extremal from a sufficiently high jet of the geodesic

All sR normal geodesics γ starting at a given point $q \in M$ with $||\dot{\gamma}(0)|| = 1$ are “parametrized” by points of the cylinder $h_g^{-1}(1/2) \cap T_q^* M$:



In different directions of D sR geodesics may be distinguished by jets of different order. For an even contact distribution there is a special (characteristic) direction C s. t. all geodesics γ with the same initial $\dot{\gamma}(0)$ not in this direction are distinguished by the 2nd jet, but the 2nd jet of all geodesics with $\dot{\gamma}(0)$ in the direction of C coincide.

The idea of the proof (continued)

Lemma

If distribution D is bracket-generating, then for a suficiently large k in a neighborhood of a generic points in T^*M the natural map

$$P_g^k \ : \ \lambda \in h_g^{-1}(1/2) \longmapsto j_0^k\left(\pi(e^{t\vec{h}_g}\lambda)\right) \qquad (k\text{-jet of } \gamma \text{ at } t=0)$$

is a diffeomorphism on its image.

The idea of the proof (continued)

Lemma

If distribution D is bracket-generating, then for a suficiently large k in a neighborhood of a generic points in T^*M the natural map

$$P_g^k \ : \ \lambda \in h_g^{-1}(1/2) \longmapsto j_0^k\left(\pi(e^{t\vec{h}_g}\lambda)\right) \qquad \text{(k-jet of γ at $t=0$)}$$

is a diffeomorphism on its image.

Lessons from the Riemannian case (Levi-Civita, Dini)

For a non rigid Riemannian metric g on M:

- **Integrability property:** The flow of normal sR extremals (of the vector field $\vec{h}_g$) admits at least one nontrivial (i.e. different from the a constant multiple of h_g) first-integrals which is quadratic on the fibers, namely the integral of (Painlevè type): if $\widetilde{g}$ is the metric projectively equivalent to g and $\{\lambda_i\}_{i=1}^m$ is the spectrum of the transition operator between g and $\widetilde{g}$, then

$$\Big(\prod_{s=1}^{m}\lambda_s\Big)^{-\frac{2}{m+1}}h_{\tilde{g}}$$

.

- **Product structure/separation of variables:** Locally $M = M_1 \times M_2$ and $g = g_1 \times g_2$ for the affine equivalence or a sort of twisted product in the case of projective equivalence.

Lessons from the Riemannian case (Levi-Civita, Dini)

For a non rigid Riemannian metric g on M:

- **Integrability property:** The flow of normal sR extremals (of the vector field $\vec{h}_g$) admits at least one nontrivial (i.e. different from the a constant multiple of h_g) first-integrals which is quadratic on the fibers, namely the integral of (Painlevè type): if $\widetilde{g}$ is the metric projectively equivalent to g and $\{\lambda_i\}_{i=1}^m$ is the spectrum of the transition operator between g and $\widetilde{g}$, then

$$\Big(\prod_{s=1}^{m} \lambda_s\Big)^{-\frac{2}{m+1}} h_{\tilde{g}}$$

.

- **Product structure/separation of variables:** Locally $M = M_1 \times M_2$ and $g = g_1 \times g_2$ for the affine equivalence or a sort of twisted product in the case of projective equivalence.

Existence of the first integral and generic projective rigidity

Definition

A sR metric g_1 is called conformally projectively rigid if $g_2 \overset{p}{\sim} g_1$ implies that g_2 is conformal to g_1.

Conformally projectively rigidity $\Rightarrow$ affine rigidity;

Theorem (Geom. Dedicata, 2019, arXiv:1801.04257v2)

If a sub-Riemannian metric is not conformally rigid, then the flow of its normal extremals admits a nontrivial integral quadratic in impulses (i.e. on the fibers of T^*M), namely the integral of Painlevè type.

Corollary

Generic sub-Riemannian metrics on a given distribution are conformally projectively rigid and therefore affinely rigid (and actually projectively rigid in real analytic category by 2020 preprint , arXiv:2001.08584).

Existence of the first integral and generic projective rigidity

Definition

A sR metric g_1 is called conformally projectively rigid if $g_2 \overset{p}{\sim} g_1$ implies that g_2 is conformal to g_1.

Conformally projectively rigidity $\Rightarrow$ affine rigidity;

Theorem (Geom. Dedicata, 2019, arXiv:1801.04257v2)

If a sub-Riemannian metric is not conformally rigid, then the flow of its normal extremals admits a nontrivial integral quadratic in impulses (i.e. on the fibers of T^*M), namely the integral of Painlevè type.

Corollary

Generic sub-Riemannian metrics on a given distribution are conformally projectively rigid and therefore affinely rigid (and actually projectively rigid in real analytic category by 2020 preprint , arXiv:2001.08584).

Existence of the first integral and generic projective rigidity

Definition

A sR metric g_1 is called conformally projectively rigid if $g_2 \overset{p}{\sim} g_1$ implies that g_2 is conformal to g_1.

Conformally projectively rigidity $\Rightarrow$ affine rigidity;

Theorem (Geom. Dedicata, 2019, arXiv:1801.04257v2)

If a sub-Riemannian metric is not conformally rigid, then the flow of its normal extremals admits a nontrivial integral quadratic in impulses (i.e. on the fibers of T^*M), namely the integral of Painlevè type.

Corollary

Generic sub-Riemannian metrics on a given distribution are conformally projectively rigid and therefore affinely rigid (and actually projectively rigid in real analytic category by 2020 preprint , arXiv:2001.08584).

Existence of the first integral and generic projective rigidity

Definition

A sR metric g_1 is called conformally projectively rigid if $g_2 \overset{p}{\sim} g_1$ implies that g_2 is conformal to g_1.

Conformally projectively rigidity $\Rightarrow$ affine rigidity;

Theorem (Geom. Dedicata, 2019, arXiv:1801.04257v2)

If a sub-Riemannian metric is not conformally rigid, then the flow of its normal extremals admits a nontrivial integral quadratic in impulses (i.e. on the fibers of T^*M), namely the integral of Painlevè type.

Corollary

Generic sub-Riemannian metrics on a given distribution are conformally projectively rigid and therefore affinely rigid (and actually projectively rigid in real analytic category by 2020 preprint , arXiv:2001.08584).

Distributions admitting product structure

Construction of pairs of projectively equivalent sub-Riemannian metrics by analogy with the metrics appearing in the Levi-Civita theorem:

Let $n = \dim M$. Fix positive integers $k_1, k_2, \ldots k_m$ such that $n = k_1 + k_2 + \ldots + k_m$. Let $\bar{x}_s = (x_s^1, \ldots, x_s^{k_s})$ and $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m)$ are standard coordinates in $\mathbb{R}^n = \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \mathbb{R}^{k_m}$, where $\mathbb{R}^{k_s}$ has standard coordinates $\bar{x}_s$.

For any $1 \le s \le m$ let D_s be a bracket generating distribution in $\mathbb{R}^{k_s}$.

Consider the distribution D on $\mathbb{R}^n$ which is obtained by the product of distributions D_s.

Definition

We will say that a distribution admits a product structure, if it is locally equivalent to such distribution D with $m \ge 2$.

Distributions admitting product structure

Construction of pairs of projectively equivalent sub-Riemannian metrics by analogy with the metrics appearing in the Levi-Civita theorem:

Let $n = \dim M$. Fix positive integers $k_1, k_2, \dots k_m$ such that $n = k_1 + k_2 + \dots + k_m$. Let $\bar{x}_s = (x_s^1, \dots, x_s^{k_s})$ and $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m)$ are standard coordinates in $\mathbb{R}^n = \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \dots \mathbb{R}^{k_m}$, where $\mathbb{R}^{k_s}$ has standard coordinates $\bar{x}_s$.

For any $1 \leq s \leq m$ let D_s be a bracket generating distribution in $\mathbb{R}^{k_s}$.

Consider the distribution D on $\mathbb{R}^n$ which is obtained by the product of distributions D_s.

Definition

We will say that a distribution admits a product structure, if it is locally equivalent to such distribution D with $m \geq 2$.

Distributions admitting product structure

Construction of pairs of projectively equivalent sub-Riemannian metrics by analogy with the metrics appearing in the Levi-Civita theorem:

Let $n = \dim M$. Fix positive integers $k_1, k_2, \ldots k_m$ such that $n = k_1 + k_2 + \ldots + k_m$. Let $\bar{x}_s = (x_s^1, \ldots, x_s^{k_s})$ and $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m)$ are standard coordinates in $\mathbb{R}^n = \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \mathbb{R}^{k_m}$, where $\mathbb{R}^{k_s}$ has standard coordinates $\bar{x}_s$.

For any $1 \leq s \leq m$ let D_s be a bracket generating distribution in $\mathbb{R}^{k_s}$.

Consider the distribution D on $\mathbb{R}^n$ which is obtained by the product of distributions D_s.

Definition

We will say that a distribution admits a product structure, if it is locally equivalent to such distribution D with $m \geq 2$.

Distributions admitting product structure

Construction of pairs of projectively equivalent sub-Riemannian metrics by analogy with the metrics appearing in the Levi-Civita theorem:

Let $n = \dim M$. Fix positive integers $k_1, k_2, \dots k_m$ such that $n = k_1 + k_2 + \dots + k_m$. Let $\bar{x}_s = (x_s^1, \dots, x_s^{k_s})$ and $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m)$ are standard coordinates in $\mathbb{R}^n = \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \dots \mathbb{R}^{k_m}$, where $\mathbb{R}^{k_s}$ has standard coordinates $\bar{x}_s$.

For any $1 \le s \le m$ let D_s be a bracket generating distribution in $\mathbb{R}^{k_s}$.

Consider the distribution D on $\mathbb{R}^n$ which is obtained by the product of distributions D_s.

Definition

We will say that a distribution admits a product structure, if it is locally equivalent to such distribution D with $m \ge 2$.

Distributions admitting product structure

Construction of pairs of projectively equivalent sub-Riemannian metrics by analogy with the metrics appearing in the Levi-Civita theorem:

Let $n = \dim M$. Fix positive integers $k_1, k_2, \dots k_m$ such that $n = k_1 + k_2 + \dots + k_m$. Let $\bar{x}_s = (x_s^1, \dots, x_s^{k_s})$ and $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m)$ are standard coordinates in $\mathbb{R}^n = \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \dots \mathbb{R}^{k_m}$, where $\mathbb{R}^{k_s}$ has standard coordinates $\bar{x}_s$.

For any $1 \leq s \leq m$ let D_s be a bracket generating distribution in $\mathbb{R}^{k_s}$.

Consider the distribution D on $\mathbb{R}^n$ which is obtained by the product of distributions D_s.

Definition

We will say that a distribution admits a product structure, if it is locally equivalent to such distribution D with $m \geq 2$.

Distributions admitting product structure

Construction of pairs of projectively equivalent sub-Riemannian metrics by analogy with the metrics appearing in the Levi-Civita theorem:

Let $n = \dim M$. Fix positive integers $k_1, k_2, \dots k_m$ such that $n = k_1 + k_2 + \dots + k_m$. Let $\bar{x}_s = (x_s^1, \dots, x_s^{k_s})$ and $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m)$ are standard coordinates in $\mathbb{R}^n = \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \dots \mathbb{R}^{k_m}$, where $\mathbb{R}^{k_s}$ has standard coordinates $\bar{x}_s$.

For any $1 \leq s \leq m$ let D_s be a bracket generating distribution in $\mathbb{R}^{k_s}$.

Consider the distribution D on $\mathbb{R}^n$ which is obtained by the product of distributions D_s.

Definition

We will say that a distribution admits a product structure, if it is locally equivalent to such distribution D with $m \geq 2$.

Examples: contact, even-contact, free distributions

Assume that D is a corank 1 distribution and α is its defining 1-form, i.e. a everywehere non-zero form annihilating D.

- D is called contact if $\mathrm{rank} D$ is even and the form $d\alpha|_D$ is nondegenerate;
- D is called even (or quasi) -contact if $\mathrm{rank} D$ is odd and $d\alpha|_D$ is one -dimensional kernel (i.e. the kernel of minimal possible dimension)

Then

- If D is contact, then it does not admit a product structure, because otherwise one of the components must be involutive and belong to the kernel of $d\alpha|_D$;
- If D is even-contact, then it admits the product structure: it is locally the product of a contact distirbution and $\mathbb{R}$;
- Free distributions (i.e. the left-invariant ones on free truncated Lie group) do not admit the product structure.

Examples: contact, even-contact, free distributions

Assume that D is a corank 1 distribution and α is its defining 1-form, i.e. a everywehere non-zero form annihilating D.

- D is called contact if $\mathrm{rank} D$ is even and the form $d\alpha|_D$ is nondegenerate;
- D is called even (or quasi) -contact if $\mathrm{rank} D$ is odd and $d\alpha|_D$ is one -dimensional kernel (i.e. the kernel of minimal possible dimension)

Then

- If D is contact, then it does not admit a product structure, because otherwise one of the components must be involutive and belong to the kernel of $d\alpha|_D$;
- If D is even-contact, then it admits the product structure: it is locally the product of a contact distirbution and $\mathbb{R}$;
- Free distributions (i.e. the left-invariant ones on free truncated Lie group) do not admit the product structure.

Examples: contact, even-contact, free distributions

Assume that D is a corank 1 distribution and α is its defining 1-form, i.e. a everywehere non-zero form annihilating D.

- D is called contact if $\mathrm{rank} D$ is even and the form $d\alpha|_D$ is nondegenerate;
- D is called even (or quasi) -contact if $\mathrm{rank} D$ is odd and $d\alpha|_D$ is one -dimensional kernel (i.e. the kernel of minimal possible dimension)

Then

- If D is contact, then it does not admit a product structure, because otherwise one of the components must be involutive and belong to the kernel of $d\alpha|_D$;
- If D is even-contact, then it admits the product structure: it is locally the product of a contact distirbution and $\mathbb{R}$;
- Free distributions (i.e. the left-invariant ones on free truncated Lie group) do not admit the product structure.

Examples: contact, even-contact, free distributions

Assume that D is a corank 1 distribution and α is its defining 1-form, i.e. a everywehere non-zero form annihilating D.

- D is called contact if $\mathrm{rank} D$ is even and the form $d\alpha|_D$ is nondegenerate;
- D is called even (or quasi) -contact if $\mathrm{rank} D$ is odd and $d\alpha|_D$ is one -dimensional kernel (i.e. the kernel of minimal possible dimension)

Then

- If D is contact, then it does not admit a product structure, because otherwise one of the components must be involutive and belong to the kernel of $d\alpha|_D$;
- If D is even-contact, then it admits the product structure: it is locally the product of a contact distirbution and $\mathbb{R}$;
- Free distributions (i.e. the left-invariant ones on free truncated Lie group) do not admit the product structure.

Examples: contact, even-contact, free distributions

Assume that D is a corank 1 distribution and α is its defining 1-form, i.e. a everywehere non-zero form annihilating D.

- D is called contact if $\mathrm{rank} D$ is even and the form $d\alpha|_D$ is nondegenerate;
- D is called even (or quasi) -contact if $\mathrm{rank} D$ is odd and $d\alpha|_D$ is one -dimensional kernel (i.e. the kernel of minimal possible dimension)

Then

- If D is contact, then it does not admit a product structure, because otherwise one of the components must be involutive and belong to the kernel of $d\alpha|_D$;
- If D is even-contact, then it admits the product structure: it is locally the product of a contact distirbution and $\mathbb{R}$;
- Free distributions (i.e. the left-invariant ones on free truncated Lie group) do not admit the product structure.

Generalized sub-Riemannian Levi-Civita pairs.

For every s, $1 \leq s \leq m$ choose a sub-Riemannian metric b_s on the distribution D_s of $\mathbb{R}^{k_s}$ and a function β_s depending on variables $\bar{x}_s$ only such that β_s is constant if $k_s > 1$ and $\beta_s(0) \neq \beta_l(0)$ for $s \neq l$.
Let

$$g_1(\dot{\bar{x}}, \dot{\bar{x}}) = \sum_{s=1}^{m} \gamma_s(\bar{x}) b_s(\dot{\bar{x}}_s, \dot{\bar{x}}_s),$$

$$g_2(\dot{\bar{x}}, \dot{\bar{x}}) = \sum_{s=1}^{m} \lambda_s(\bar{x}) \gamma_s(\bar{x}) b_s(\dot{\bar{x}}_s, \dot{\bar{x}}_s)$$

where the velocities $\dot{\bar{x}}$ belong to D, $\lambda_s(\bar{x}) = \beta_s(\bar{x}_s) \prod_{l=1}^{m} \beta_l(\bar{x}_l)$,
$\gamma_s(\bar{x}) = \prod_{l \neq s} \left| \frac{1}{\beta_l(\bar{x}_l)} - \frac{1}{\beta_s(\bar{x}_s)} \right|$.
Then $g_1 \overset{p}{\sim} g_2$ near the origin.

Also, the normal extremal flow of g_1 admits m integrals in involution as in the Riemannian case.

Generalized sub-Riemannian Levi-Civita pairs.

For every s, $1 \leq s \leq m$ choose a sub-Riemannian metric b_s on the distribution D_s of $\mathbb{R}^{k_s}$ and a function β_s depending on variables $\bar{x}_s$ only such that β_s is constant if $k_s > 1$ and $\beta_s(0) \neq \beta_l(0)$ for $s \neq l$.
Let

$$g_1(\dot{\bar{x}}, \dot{\bar{x}}) = \sum_{s=1}^{m} \gamma_s(\bar{x}) b_s(\dot{\bar{x}}_s, \dot{\bar{x}}_s),$$

$$g_2(\dot{\bar{x}}, \dot{\bar{x}}) = \sum_{s=1}^{m} \lambda_s(\bar{x}) \gamma_s(\bar{x}) b_s(\dot{\bar{x}}_s, \dot{\bar{x}}_s)$$

where the velocities $\dot{\bar{x}}$ belong to D, $\lambda_s(\bar{x}) = \beta_s(\bar{x}_s) \prod_{l=1}^{m} \beta_l(\bar{x}_l)$, $\gamma_s(\bar{x}) = \prod_{l \neq s} \left| \frac{1}{\beta_l(\bar{x}_l)} - \frac{1}{\beta_s(\bar{x}_s)} \right|$.
Then $g_1 \overset{p}{\sim} g_2$ near the origin.

Also, the normal extremal flow of g_1 admits m integrals in involution as in the Riemannian case.

Generalized sub-Riemannian Levi-Civita pairs.

For every s, $1 \leq s \leq m$ choose a sub-Riemannian metric b_s on the distribution D_s of $\mathbb{R}^{k_s}$ and a function β_s depending on variables $\bar{x}_s$ only such that β_s is constant if $k_s > 1$ and $\beta_s(0) \neq \beta_l(0)$ for $s \neq l$.

Let

$$g_1(\dot{\bar{x}}, \dot{\bar{x}}) = \sum_{s=1}^{m} \gamma_s(\bar{x}) b_s(\dot{\bar{x}}_s, \dot{\bar{x}}_s),$$

$$g_2(\dot{\bar{x}}, \dot{\bar{x}}) = \sum_{s=1}^{m} \lambda_s(\bar{x}) \gamma_s(\bar{x}) b_s(\dot{\bar{x}}_s, \dot{\bar{x}}_s)$$

where the velocities $\dot{\bar{x}}$ belong to D, $\lambda_s(\bar{x}) = \beta_s(\bar{x}_s) \prod_{l=1}^{m} \beta_l(\bar{x}_l)$, $\gamma_s(\bar{x}) = \prod_{l \neq s} \left| \frac{1}{\beta_l(\bar{x}_l)} - \frac{1}{\beta_s(\bar{x}_s)} \right|$.

Then $g_1 \overset{p}{\sim} g_2$ near the origin.

Also, the normal extremal flow of g_1 admits m integrals in involution as in the Riemannian case.

Generalized sub-Riemannian Levi-Civita pairs.

For every s, $1 \le s \le m$ choose a sub-Riemannian metric b_s on the distribution D_s of $\mathbb{R}^{k_s}$ and a function β_s depending on variables $\bar{x}_s$ only such that β_s is constant if $k_s > 1$ and $\beta_s(0) \neq \beta_l(0)$ for $s \neq l$.
Let

$$g_1(\dot{\bar{x}}, \dot{\bar{x}}) = \sum_{s=1}^{m} \gamma_s(\bar{x}) b_s(\dot{\bar{x}}_s, \dot{\bar{x}}_s),$$

$$g_2(\dot{\bar{x}}, \dot{\bar{x}}) = \sum_{s=1}^{m} \lambda_s(\bar{x}) \gamma_s(\bar{x}) b_s(\dot{\bar{x}}_s, \dot{\bar{x}}_s)$$

where the velocities $\dot{\bar{x}}$ belong to D, $\lambda_s(\bar{x}) = \beta_s(\bar{x}_s) \prod_{l=1}^{m} \beta_l(\bar{x}_l)$,
$\gamma_s(\bar{x}) = \prod_{l \neq s} \left| \frac{1}{\beta_l(\bar{x}_l)} - \frac{1}{\beta_s(\bar{x}_s)} \right|$.

Then $g_1 \overset{p}{\sim} g_2$ near the origin.

Also, the normal extremal flow of g_1 admits m integrals in involution as in the Riemannian case.

Generalized sub-Riemannian Levi-Civita pairs.

For every s, $1 \leq s \leq m$ choose a sub-Riemannian metric b_s on the distribution D_s of $\mathbb{R}^{k_s}$ and a function β_s depending on variables $\bar{x}_s$ only such that β_s is constant if $k_s > 1$ and $\beta_s(0) \neq \beta_l(0)$ for $s \neq l$.
Let

$$g_1(\dot{\bar{x}}, \dot{\bar{x}}) = \sum_{s=1}^{m} \gamma_s(\bar{x}) b_s(\dot{\bar{x}}_s, \dot{\bar{x}}_s),$$

$$g_2(\dot{\bar{x}}, \dot{\bar{x}}) = \sum_{s=1}^{m} \lambda_s(\bar{x}) \gamma_s(\bar{x}) b_s(\dot{\bar{x}}_s, \dot{\bar{x}}_s)$$

where the velocities $\dot{\bar{x}}$ belong to D, $\lambda_s(\bar{x}) = \beta_s(\bar{x}_s) \prod_{l=1}^{m} \beta_l(\bar{x}_l)$,
$\gamma_s(\bar{x}) = \prod_{l \neq s} \left| \frac{1}{\beta_l(\bar{x}_l)} - \frac{1}{\beta_s(\bar{x}_s)} \right|$.
Then $g_1 \overset{p}{\sim} g_2$ near the origin.

Also, the normal extremal flow of g_1 admits m integrals in involution as in the Riemannian case.

Generalized sub-Riemannian Levi-Civita pairs.

For every s, $1 \leq s \leq m$ choose a sub-Riemannian metric b_s on the distribution D_s of $\mathbb{R}^{k_s}$ and a function β_s depending on variables $\bar{x}_s$ only such that β_s is constant if $k_s > 1$ and $\beta_s(0) \neq \beta_l(0)$ for $s \neq l$.
Let

$$g_1(\dot{\bar{x}}, \dot{\bar{x}}) = \sum_{s=1}^{m} \gamma_s(\bar{x}) b_s(\dot{\bar{x}}_s, \dot{\bar{x}}_s),$$

$$g_2(\dot{\bar{x}}, \dot{\bar{x}}) = \sum_{s=1}^{m} \lambda_s(\bar{x}) \gamma_s(\bar{x}) b_s(\dot{\bar{x}}_s, \dot{\bar{x}}_s)$$

where the velocities $\dot{\bar{x}}$ belong to D, $\lambda_s(\bar{x}) = \beta_s(\bar{x}_s) \prod_{l=1}^{m} \beta_l(\bar{x}_l)$,
$\gamma_s(\bar{x}) = \prod_{l \neq s} \left| \frac{1}{\beta_l(\bar{x}_l)} - \frac{1}{\beta_s(\bar{x}_s)} \right|$.
Then $g_1 \overset{p}{\sim} g_2$ near the origin.

Also, the normal extremal flow of g_1 admits m integrals in involution as in the Riemannian case.

The main conjecture

Conjecture

The generalized Levi-Civita pairs are the only pairs of locally projectively equivalent sR metrics and the generalized Levi-Civita pairs with constant β's are the only pairs of locally affinely equivalent sR metrics under certain regularity assumptions (stability of the transition operator+equiregularity of distribution)

The full positive answer beyond Riemannian metrics was obtained for corank 1 distributions only (I. Z., 2006, A. Castillo, I. Z., 2014).

The main conjecture

Conjecture

The generalized Levi-Civita pairs are the only pairs of locally projectively equivalent sR metrics and the generalized Levi-Civita pairs with constant β's are the only pairs of locally affinely equivalent sR metrics under certain regularity assumptions (stability of the transition operator+equiregularity of distribution)

The full positive answer beyond Riemannian metrics was obtained for corank 1 distributions only (I. Z., 2006, A. Castillo, I. Z., 2014).

Sub-Riemannian Weyl type results: briefly

The conjecture is also true if in addition we assume that the metrics under consideration are conformal, all objects are real analytic and (complexified) abnormal extremals of D satisfy some special properties:

In this case the conjecture says that *two conformal metrics are locally projectively equivalent if and only if they are constantly proportional.* (2020 preprint , arXiv:2001.08584).

In Riemannian geometry it is always true (for $n > 1$). This result is attributed to H. Weyl, although it is a particular case of Levi-Civita Theorem, so we call such results sub-Riemannian Weyl theorems and the metric satisfying this result Weyl rigid.

Sub-Riemannian Weyl type results: briefly

The conjecture is also true if in addition we assume that the metrics under consideration are conformal, all objects are real analytic and (complexified) abnormal extremals of D satisfy some special properties:

In this case the conjecture says that *two conformal metrics are locally projectively equivalent if and only if they are constantly proportional.* (2020 preprint , arXiv:2001.08584).

In Riemannian geometry it is always true (for $n > 1$). This result is attributed to H. Weyl, although it is a particular case of Levi-Civita Theorem, so we call such results sub-Riemannian Weyl theorems and the metric satisfying this result Weyl rigid.

Sub-Riemannian Weyl type results: briefly

The conjecture is also true if in addition we assume that the metrics under consideration are conformal, all objects are real analytic and (complexified) abnormal extremals of D satisfy some special properties:

In this case the conjecture says that *two conformal metrics are locally projectively equivalent if and only if they are constantly proportional.* (2020 preprint , arXiv:2001.08584).

In Riemannian geometry it is always true (for $n > 1$). This result is attributed to H. Weyl, although it is a particular case of Levi-Civita Theorem, so we call such results sub-Riemannian Weyl theorems and the metric satisfying this result Weyl rigid.

Weaker separation results

If the Conjecture is true then it establish the separation/product structure for the distribution (if the metric is not conformally rigid, and also for the metric (at least in the cae of affine equivalence or a twisted vversion fo it in the cae of projective equivalence).

We established two weaker separation results:

- Separation on the level of the nilpotent approximation of the sR metrics in projective case;
- Separation on the level of Jacobi curves along generic extremals (decoupling of the Jacobi equation) in the case fo affine equivalence but for more general than sub-Riemannian (sub-Finslerian, affine) problems.

Weaker separation results

If the Conjecture is true then it establish the separation/product structure for the distribution (if the metric is not conformally rigid, and also for the metric (at least in the cae of affine equivalence or a twisted vversion fo it in the cae of projective equivalence).

We established two weaker separation results:

- Separation on the level of the nilpotent approximation of the sR metrics in projective case;
- Separation on the level of Jacobi curves along generic extremals (decoupling of the Jacobi equation) in the case fo affine equivalence but for more general than sub-Riemannian (sub-Finslerian, affine) problems.

Weaker separation results

If the Conjecture is true then it establish the separation/product structure for the distribution (if the metric is not conformally rigid, and also for the metric (at least in the cae of affine equivalence or a twisted vversion fo it in the cae of projective equivalence).
We established two weaker separation results:

- Separation on the level of the nilpotent approximation of the sR metrics in projective case;
- Separation on the level of Jacobi curves along generic extremals (decoupling of the Jacobi equation) in the case fo affine equivalence but for more general than sub-Riemannian (sub-Finslerian, affine) problems.

Tanaka symbol and nilpotent approximation of a distribution

D is called equiregular at q_0 if all D^j have constant dimension in a neighborhood of q_0.

Definition

- *The (Tanaka) symbol of an equiregular distribution D at a point q_0 is the graded nilpotent Lie algebra*

$$\underbrace{D(q_0)}_{\mathfrak{g}_{-1}(q_0)} \oplus \underbrace{D^2(q_0)/D(q_0)}_{\mathfrak{g}_{-2}(q_0)} \oplus \underbrace{D^3(q_0)/D^2(q_0)}_{\mathfrak{g}_{-3}(q_0)} \oplus \cdots .$$

- *The left-invariant distribution on the corresponding Lie group obtained by the left translation of $D(q_0)$ is called the nilpotent approximation of D at q_0 and is denote by $\widehat{D}_{q_0}$.*

Tanaka symbol and nilpotent approximation of a distribution

D is called equiregular at q_0 if all D^j have constant dimension in a neighborhood of q_0.

Definition

- *The (Tanaka) symbol of an equiregular distribution D at a point q_0 is the graded nilpotent Lie algebra*

$$\underbrace{D(q_0)}_{\mathfrak{g}_{-1}(q_0)} \oplus \underbrace{D^2(q_0)/D(q_0)}_{\mathfrak{g}_{-2}(q_0)} \oplus \underbrace{D^3(q_0)/D^2(q_0)}_{\mathfrak{g}_{-3}(q_0)} \oplus \cdots .$$

- *The left-invariant distribution on the corresponding Lie group obtained by the left translation of $D(q_0)$ is called the nilpotent approximation of D at q_0 and is denote by $\widehat{D}_{q_0}$.*

Tanaka symbol and nilpotent approximation of a distribution

D is called equiregular at q_0 if all D^j have constant dimension in a neighborhood of q_0.

Definition

- *The (Tanaka) symbol of an equiregular distribution D at a point q_0 is the graded nilpotent Lie algebra*

$$\underbrace{D(q_0)}_{\mathfrak{g}_{-1}(q_0)} \oplus \underbrace{D^2(q_0)/D(q_0)}_{\mathfrak{g}_{-2}(q_0)} \oplus \underbrace{D^3(q_0)/D^2(q_0)}_{\mathfrak{g}_{-3}(q_0)} \oplus \cdots .$$

- *The left-invariant distribution on the corresponding Lie group obtained by the left translation of $D(q_0)$ is called the nilpotent approximation of D at q_0 and is denote by $\widehat{D}_{q_0}$.*

Tanaka symbol and nilpotent approximation of a distribution

D is called equiregular at q_0 if all D^j have constant dimension in a neighborhood of q_0.

Definition

- *The (Tanaka) symbol of an equiregular distribution D at a point q_0 is the graded nilpotent Lie algebra*

$$\underbrace{D(q_0)}_{\mathfrak{g}_{-1}(q_0)} \oplus \underbrace{D^2(q_0)/D(q_0)}_{\mathfrak{g}_{-2}(q_0)} \oplus \underbrace{D^3(q_0)/D^2(q_0)}_{\mathfrak{g}_{-3}(q_0)} \oplus \cdots .$$

- *The left-invariant distribution on the corresponding Lie group obtained by the left translation of $D(q_0)$ is called the nilpotent approximation of D at q_0 and is denote by $\widehat{D}_{q_0}$.*

Example: Tanaka symbol of contact distributions

For example, if D is a contact distribution of rank $2n$, then its Tanaka symbol is isomorphic to the $2n+1$ dimensional Heisenberg algebra:

$$(X,Y) \mapsto [X,Y]$$

defines a simplectic form σ on D, up to a multiplication by a constant , corresponding to the choice of the basis vector Z of D^2/D).

$$[X,Y] = \sigma(X,Y)Z$$

Take the Darboux basis $E_1,\dots,E_n,F_i,\dots F_n$ of D with respect to σ, i.e. such that $\sigma(E_i,F_j)=\delta_{ij}$.
Then $[E_j,F_j]=\delta_{ij}Z$ and it is the standard basis in the Heisenberg algebra.

Example: Tanaka symbol of contact distributions

For example, if D is a contact distribution of rank $2n$, then its Tanaka symbol is isomorphic to the $2n+1$ dimensional Heisenberg algebra:

$$(X, Y) \mapsto [X, Y]$$

defines a simplectic form σ on D, up to a multiplication by a constant , corresponding to the choice of the basis vector Z of D^2/D).

$$[X, Y] = \sigma(X, Y)Z$$

Take the Darboux basis $E_1, \dots, E_n, F_i, \dots F_n$ of D with respect to σ, i.e. such that $\sigma(E_i, F_j) = \delta_{ij}$.
Then $[E_j, F_j] = \delta_{ij} Z$ and it is the standard basis in the Heisenberg algebra.

Definition

- *The symbol of an sR metric g is the pair consisting of the Tanaka symbol of D at q_0 and the Euclidean structure $g(q_0)$ on $D(q_0)$.*
- *The nilpotent approximation of sub-Riemannian metric g on an equiregular distribution D at a point q_0 is the left-invariant sR structure $\hat{g}$ on the Lie group of the Tanaka symbol of D at q_0 such that the Euclidean structure at the identity coincides with the Euclidean structure at $D(q_0)$.*

Symbol and Nilpotent approximation of a sR structure

Definition

- *The symbol of an sR metric g is the pair consisting of the Tanaka symbol of D at q_0 and the Euclidean structure $g(q_0)$ on $D(q_0)$.*
- *The nilpotent approximation of sub-Riemannian metric g on an equiregular distribution D at a point q_0 is the left-invariant sR structure $\hat{g}$ on the Lie group of the Tanaka symbol of D at q_0 such that the Euclidean structure at the identity coincides with the Euclidean structure at $D(q_0)$.*

Direct product structure on the level of nilpotent approximation

Theorem (Geom. Dedicata, 2019, arXiv:1801.04257v2)

If g_1 and g_2 are two sub-Riemannian metric on an equiregular distribution D, which are locally projectively equivalent around a stable point q_0 and not conformal, then the nilpotent approximation $\hat{D}_{q_0}$ of D at q_0 admits a product structure and the corresponding nilpotent approximations $\hat{g}_1$ and $\hat{g}_2$ form a Levi-Civita pair with constant coefficients.

Corollary

Any sub-Riemannian metric on a rank 2 bracket generating distribution is affinely rigid and conformally projectively rigid.

Direct product structure on the level of nilpotent approximation

Theorem (Geom. Dedicata, 2019, arXiv:1801.04257v2)

If g_1 and g_2 are two sub-Riemannian metric on an equiregular distribution D, which are locally projectively equivalent around a stable point q_0 and not conformal, then the nilpotent approximation $\hat{D}_{q_0}$ of D at q_0 admits a product structure and the corresponding nilpotent approximations $\hat{g}_1$ and $\hat{g}_2$ form a Levi-Civita pair with constant coefficients.

Corollary

Any sub-Riemannian metric on a rank 2 bracket generating distribution is affinely rigid and conformally projectively rigid.

Genericity of indecomposable fundamental graded Lie algebras

Let $\mathrm{GNLA}(m,n)$ be the set of all n-dimensional negatively graded Lie algebras generated by the homogeneous component of weight -1 and such that this component has dimension m.

Proposition

Except the following two cases:

1. $m = n - 1$ *with even* n,
2. $(m,n) = (4,6)$,

a generic element of $\mathrm{GNLA}(m,n)$ *cannot be represented as a direct sum of two graded Lie algebras.*

Genericity of indecomposable fundamental graded Lie algebras

Let $\mathrm{GNLA}(m, n)$ be the set of all n-dimensional negatively graded Lie algebras generated by the homogeneous component of weight -1 and such that this component has dimension m.

Proposition

Except the following two cases:

1. $m = n - 1$ *with even* n,
2. $(m, n) = (4, 6)$,

a generic element of $\mathrm{GNLA}(m, n)$ *cannot be represented as a direct sum of two graded Lie algebras.*

Rigidity of SR structures on generic distribution

Theorem (Geom. Dedicata, 2019, arXiv:1801.04257v2)

Let m and n be two integers such that $2 \leq m < n$, and assume $(m, n) \neq (4, 6)$ and $m \neq n - 1$ if n is even. Then, given an n-dimensional manifold M and a generic rank m distribution D on M, any sub-Riemannian metric on (M, D) conformally projectively rigid and therefore affinely rigid (and in the real analytic category even projectively rigid from the following sub-Riemannian Weyl results).

Theorem (preprint, arXiv:2001.08584)

Let m and n be two integers such that $2 \leq m < n$. On a generic real analytic rank m distribution D on a connected n-dimensional real analytic manifold M any sub-Riemannian metric is Weyl projectively rigid.

Rigidity of SR structures on generic distribution

Theorem (Geom. Dedicata, 2019, arXiv:1801.04257v2)

Let m and n be two integers such that $2 \leq m < n$, and assume $(m, n) \neq (4, 6)$ and $m \neq n - 1$ if n is even. Then, given an n-dimensional manifold M and a generic rank m distribution D on M, any sub-Riemannian metric on (M, D) conformally projectively rigid and therefore affinely rigid (and in the real analytic category even projectively rigid from the following sub-Riemannian Weyl results).

Theorem (preprint, arXiv:2001.08584)

Let m and n be two integers such that $2 \leq m < n$. On a generic real analytic rank m distribution D on a connected n-dimensional real analytic manifold M any sub-Riemannian metric is Weyl projectively rigid.

Decomposible in terms of spaces of skew-symmetric forms

If D is of step 2, i.e. when $D^2 = TM$, then the Tanaka symbol is described by the the *Levi operator* $\mathcal{L} : \wedge^2 D \mapsto D^2/D (\cong TM/D)$ or , equivalently, by the dual operator $\mathcal{L} : D^* \mapsto \wedge^2 D^*$.

The image of this operator is the $(n-m)$-dimensional subspace Ω in the space of skew-symmetric forms on D.

The Tanaka symbol is decomposible if and only $Omega_{\mathfrak{g}} = \Omega^1_{\mathfrak{g}} \oplus \Omega^2_{\mathfrak{g}}$ s.t. in some basis of $d = \mathfrak{g}_{-1}$, the elements of $\Omega^1_{\mathfrak{g}}$ are $\left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & 0 \end{array}\right)$ and the elements of $\Omega^2_{\mathfrak{g}}$ are $\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A_2 \end{array}\right)$, where the corresponding blocks have the same nonzero size.

Decomposible in terms of spaces of skew-symmetric forms

If D is of step 2, i.e. when $D^2 = TM$, then the Tanaka symbol is described by the the *Levi operator* $\mathcal{L} : \wedge^2 D \mapsto D^2/D (\cong TM/D)$ or , equivalently, by the dual operator $\mathcal{L} : D^* \mapsto \wedge^2 D^*$.

The image of this operator is the $(n-m)$-dimensional subspace Ω in the space of skew-symmetric forms on D.

The Tanaka symbol is decomposible if and only $Omega_{\mathfrak{g}} = \Omega^1_{\mathfrak{g}} \oplus \Omega^2_{\mathfrak{g}}$ s.t. in some basis of $d = \mathfrak{g}_{-1}$, the elements of $\Omega^1_{\mathfrak{g}}$ are $\left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & 0 \end{array}\right)$ and the elements of $\Omega^2_{\mathfrak{g}}$ are $\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A_2 \end{array}\right)$, where the corresponding blocks have the same nonzero size.

Decomposible in terms of spaces of skew-symmetric forms

If D is of step 2, i.e. when $D^2 = TM$, then the Tanaka symbol is described by the the *Levi operator* $\mathcal{L} : \wedge^2 D \mapsto D^2/D (\cong TM/D)$ or , equivalently, by the dual operator $\mathcal{L} : D^* \mapsto \wedge^2 D^*$.

The image of this operator is the $(n-m)$-dimensional subspace Ω in the space of skew-symmetric forms on D.

The Tanaka symbol is decomposible if and only if $Omega_{\mathfrak{g}} = \Omega^1_{\mathfrak{g}} \oplus \Omega^2_{\mathfrak{g}}$ s.t. in some basis of $d = \mathfrak{g}_{-1}$, the elements of $\Omega^1_{\mathfrak{g}}$ are $\left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & 0 \end{array}\right)$ and the elements of $\Omega^2_{\mathfrak{g}}$ are $\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A_2 \end{array}\right)$, where the corresponding blocks have the same nonzero size.

Decomposible in terms of spaces of skew-symmetric forms

If D is of step 2, i.e. when $D^2 = TM$, then the Tanaka symbol is described by the the *Levi operator* $\mathcal{L} : \wedge^2 D \mapsto D^2/D (\cong TM/D)$ or , equivalently, by the dual operator $\mathcal{L} : D^* \mapsto \wedge^2 D^*$.

The image of this operator is the $(n-m)$-dimensional subspace Ω in the space of skew-symmetric forms on D.

The Tanaka symbol is decomposible if and only $Omega_{\mathfrak{g}} = \Omega^1_{\mathfrak{g}} \oplus \Omega^2_{\mathfrak{g}}$ s.t. in some basis of $d = \mathfrak{g}_{-1}$, the elements of $\Omega^1_{\mathfrak{g}}$ are $\left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & 0 \end{array}\right)$ and the elements of $\Omega^2_{\mathfrak{g}}$ are $\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A_2 \end{array}\right)$, where the corresponding blocks have the same nonzero size.

Why the Tanaka symbol in (4,6) case is not generically indecomposible?

In the case of $n - m = 2$ (i.e. corank is 2) it is a pencil (i.e. a plane) of skew-symmetric forms $\Rightarrow$ Kronecker theory of pencils.
For $(m, n) = (4, 6)$ the equation $\mathrm{Pfaffian}(\omega) = 0, \omega \in \Omega$ is quadratic.

If there are two distinguished (real) lines l_1 and l_2 in Ω satisfying this equation (an open condition), D_1 and D_2 are two planes , which are kernels of the forms on each line. $\Rightarrow$

Ω_g can be decomposed into sum of two lines of the form $\left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & 0 \end{array}\right)$ and $\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A_2 \end{array}\right)$ in the bases compatible with the splitting $D = D_1 \oplus D_2$.
Then $\widehat{D} = \widehat{D_1} \times \widehat{D_2}$, and D_i form contact $(2, 3)$ -distributions.

Why the Tanaka symbol in (4,6) case is not generically indecomposible?

In the case of $n - m = 2$ (i.e. corank is 2) it is a pencil (i.e. a plane) of skew-symmetric forms $\Rightarrow$ Kronecker theory of pencils.
For $(m, n) = (4, 6)$ the equation $\mathrm{Pfaffian}(\omega) = 0, \omega \in \Omega$ is quadratic.

If there are two distinguished (real) lines l_1 and l_2 in Ω satisfying this equation (an open condition), D_1 and D_2 are two planes , which are kernels of the forms on each line. $\Rightarrow$

Ω_g can be decomposed into sum of two lines of the form $\left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & 0 \end{array}\right)$ and $\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A_2 \end{array}\right)$ in the bases compatible with the splitting $D = D_1 \oplus D_2$.
Then $\widehat{D} = \widehat{D_1} \times \widehat{D_2}$, and D_i form contact $(2, 3)$ -distributions.

Why the Tanaka symbol in (4,6) case is not generically indecomposible?

In the case of $n-m=2$ (i.e. corank is 2) it is a pencil (i.e. a plane) of skew-symmetric forms $\Rightarrow$ Kronecker theory of pencils.
For $(m,n)=(4,6)$ the equation $\mathrm{Pfaffian}(\omega)=0, \omega\in\Omega$ is quadratic.

If there are two distinguished (real) lines l_1 and l_2 in Ω satisfying this equation (an open condition), D_1 and D_2 are two planes , which are kernels of the forms on each line. $\Rightarrow$

Ω_g can be decomposed into sum of two lines of the form $\left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & 0 \end{array}\right)$ and $\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A_2 \end{array}\right)$ in the bases compatible with the splitting $D=D_1\oplus D_2$.
Then $\widehat{D}=\widehat{D_1}\times\widehat{D_2}$, and D_i form contact $(2,3)$ -distributions.

Why the Tanaka symbol in (4,6) case is not generically indecomposible?

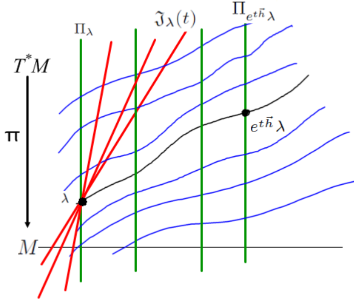
In the case of $n - m = 2$ (i.e. corank is 2) it is a pencil (i.e. a plane) of skew-symmetric forms $\Rightarrow$ Kronecker theory of pencils.
For $(m, n) = (4, 6)$ the equation $\mathrm{Pfaffian}(\omega) = 0, \omega \in \Omega$ is quadratic.

If there are two distinguished (real) lines l_1 and l_2 in Ω satisfying this equation (an open condition), D_1 and D_2 are two planes , which are kernels of the forms on each line. $\Rightarrow$

Ω_g can be decomposed into sum of two lines of the form $\left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & 0 \end{array}\right)$ and $\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A_2 \end{array}\right)$ in the bases compatible with the splitting $D = D_1 \oplus D_2$.
Then $\widehat{D} = \widehat{D_1} \times \widehat{D_2}$, and D_i form contact $(2, 3)$ -distributions.

Jacobi curves of normal extremals

Let Π_λ be the vertical subspace of $T_\lambda T^*M$, i.e. the tangent to the fiber at λ:



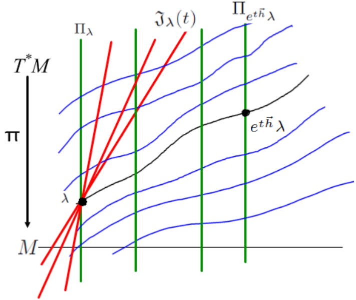
Let $h := h_g$. To any extremal $e^{t\vec{h}}\lambda$ assign the curve of Lagrangian subspaces

$$t \longmapsto \mathfrak{J}_\lambda(t) := d(e^{-t\vec{h}})(\Pi_{e^{t\vec{h}}\lambda})$$

in the symplectic space $T_\lambda T^*M$, the Jacobi curve of the extremal $e^{t\vec{h}}\lambda$.

Jacobi curves of normal extremals

Let Π_λ be the vertical subspace of $T_\lambda T^*M$, i.e. the tangent to the fiber at λ:



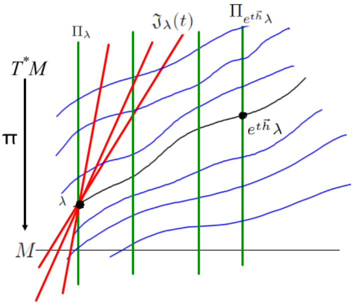
Let $h := h_g$. To any extremal $e^{t\vec{h}}\lambda$ assign the curve of Lagrangian subspaces

$$t \longmapsto \mathfrak{J}_\lambda(t) := d(e^{-t\vec{h}})(\Pi_{e^{t\vec{h}}\lambda})$$

in the symplectic space $T_\lambda T^*M$, the Jacobi curve of the extremal $e^{t\vec{h}}\lambda$.

Jacobi curves of normal extremals

Let Π_λ be the vertical subspace of $T_\lambda T^*M$, i.e. the tangent to the fiber at λ:



Let $h := h_g$. To any extremal $e^{t\vec{h}}\lambda$ assign the curve of Lagrangian subspaces

$$t \longmapsto \mathfrak{J}_\lambda(t) := d(e^{-t\vec{h}})(\Pi_{e^{t\vec{h}}\lambda})$$

in the symplectic space $T_\lambda T^*M$, the Jacobi curve of the extremal $e^{t\vec{h}}\lambda$.

Jacobi curves: conjugate points, sub-Riemannian connection and curvature

Jacobi curves are curves in Lagrangian Grassmannians (LG).

They contain all information about Jacobi fields and conjugate points along extremals.) For example, a point $\tilde{t}$ is conjugate to 0 along the extremal $e^{t\vec{h}}\lambda$ iff

$$\mathfrak{J}_\lambda(\tilde{t}) \cap \mathfrak{J}_\lambda(0) \neq 0.$$

Any symplectic invariant of a Jacobi curve (i.e. the invariant under the action of the symplectic group on $T_\lambda T^*M$) produces a function on T^*M For example, symplectically invariant constructions with Jacobi curves of Riemannian extremals gives an alternative construction of the the Riemannian curvature tensor .

Studying more general curves in LG one can construct analogous canonical (but non-linear) Ehresmann connection and curvature type invariants for any sub-Riemannian metric and more general geometric structure (Agrachev-I.Z.(20002)., Chengbo Li -I.Z. (2009).

Jacobi curves: conjugate points, sub-Riemannian connection and curvature

Jacobi curves are curves in Lagrangian Grassmannians (LG).

They contain all information about Jacobi fields and conjugate points along extremals.) For example, a point $\tilde{t}$ is conjugate to 0 along the extremal $e^{t\vec{h}}\lambda$ iff

$$\mathfrak{J}_\lambda(\tilde{t}) \cap \mathfrak{J}_\lambda(0) \neq 0.$$

Any symplectic invariant of a Jacobi curve (i.e. the invariant under the action of the symplectic group on $T_\lambda T^*M$) produces a function on T^*M For example, symplectically invariant constructions with Jacobi curves of Riemannian extremals gives an alternative construction of the the Riemannian curvature tensor .

Studying more general curves in LG one can construct analogous canonical (but non-linear) Ehresmann connection and curvature type invariants for any sub-Riemannian metric and more general geometric structure (Agrachev-I.Z.(20002)., Chengbo Li -I.Z. (2009).

Jacobi curves: conjugate points, sub-Riemannian connection and curvature

Jacobi curves are curves in Lagrangian Grassmannians (LG).

They contain all information about Jacobi fields and conjugate points along extremals.) For example, a point $\tilde{t}$ is conjugate to 0 along the extremal $e^{t\vec{h}}\lambda$ iff

$$\mathfrak{J}_\lambda(\tilde{t}) \cap \mathfrak{J}_\lambda(0) \neq 0.$$

Any symplectic invariant of a Jacobi curve (i.e. the invariant under the action of the symplectic group on $T_\lambda T^*M$) produces a function on T^*M For example, symplectically invariant constructions with Jacobi curves of Riemannian extremals gives an alternative construction of the the Riemannian curvature tensor .

Studying more general curves in LG one can construct analogous canonical (but non-linear) Ehresmann connection and curvature type invariants for any sub-Riemannian metric and more general geometric structure (Agrachev-I.Z.(20002)., Chengbo Li -I.Z. (2009).

Jacobi curves: conjugate points, sub-Riemannian connection and curvature

Jacobi curves are curves in Lagrangian Grassmannians (LG).

They contain all information about Jacobi fields and conjugate points along extremals.) For example, a point $\tilde{t}$ is conjugate to 0 along the extremal $e^{t\vec{h}}\lambda$ iff

$$\mathfrak{J}_\lambda(\tilde{t}) \cap \mathfrak{J}_\lambda(0) \neq 0.$$

Any symplectic invariant of a Jacobi curve (i.e. the invariant under the action of the symplectic group on $T_\lambda T^*M$) produces a function on T^*M For example, symplectically invariant constructions with Jacobi curves of Riemannian extremals gives an alternative construction of the the Riemannian curvature tensor .

Studying more general curves in LG one can construct analogous canonical (but non-linear) Ehresmann connection and curvature type invariants for any sub-Riemannian metric and more general geometric structure (Agrachev-I.Z.(20002)., Chengbo Li -I.Z. (2009).

Jacobi curves: conjugate points, sub-Riemannian connection and curvature

Jacobi curves are curves in Lagrangian Grassmannians (LG).

They contain all information about Jacobi fields and conjugate points along extremals.) For example, a point $\tilde{t}$ is conjugate to 0 along the extremal $e^{t\vec{h}}\lambda$ iff

$$\mathfrak{J}_\lambda(\tilde{t}) \cap \mathfrak{J}_\lambda(0) \neq 0.$$

Any symplectic invariant of a Jacobi curve (i.e. the invariant under the action of the symplectic group on $T_\lambda T^*M$) produces a function on T^*M For example, symplectically invariant constructions with Jacobi curves of Riemannian extremals gives an alternative construction of the the Riemannian curvature tensor .

Studying more general curves in LG one can construct analogous canonical (but non-linear) Ehresmann connection and curvature type invariants for any sub-Riemannian metric and more general geometric structure (Agrachev-I.Z.(20002)., Chengbo Li -I.Z. (2009).

Separation/direct product on the level of Jacobi equations/Jacobi curves of extremals

Projective/affine equivalence of g_1 and $g_2 \Rightarrow$ existence of the fiber-preserving preserving orbital diffeomorphism Φ between Hamiltonian flows of the correspondng HAmiltonians $\vec{h}_{g_1}$ and $\vec{h}_{g_2}$ on the open dense set of $T^*M \Rightarrow$

Φ_* sends the Jacobi curve at λ of the corresponding extremal of g_1 to the Jacobi curve at $\Phi(\lambda)$ of the corresponding extremal g_2 (the curves are considered as unparametrized curves)

Theorem (I.Z.)

If a sub-Riemannian metric is not affinely rigid, then the Jacobi curve of a generic normal extremal is a direct product of curves in Lagrangian Grassmannians of smaller dimensions.

Separation/direct product on the level of Jacobi equations/Jacobi curves of extremals

Projective/affine equivalence of g_1 and g_2 $\Rightarrow$ existence of the fiber-preserving preserving orbital diffeomorphism Φ between Hamiltonian flows of the correspondng HAmiltonians $\vec{h}_{g_1}$ and $\vec{h}_{g_2}$ on the open dense set of T^*M $\Rightarrow$

Φ_* sends the Jacobi curve at λ of the corresponding extremal of g_1 to the Jacobi curve at $\Phi(\lambda)$ of the corresponding extremal g_2 (the curves are considered as unparametrized curves)

Theorem (I.Z.)

If a sub-Riemannian metric is not affinely rigid, then the Jacobi curve of a generic normal extremal is a direct product of curves in Lagrangian Grassmannians of smaller dimensions.

Separation/direct product on the level of Jacobi equations/Jacobi curves of extremals

Projective/affine equivalence of g_1 and $g_2 \Rightarrow$ existence of the fiber-preserving preserving orbital diffeomorphism Φ between Hamiltonian flows of the correspondng HAmiltonians $\vec{h}_{g_1}$ and $\vec{h}_{g_2}$ on the open dense set of $T^*M \Rightarrow$

Φ_* sends the Jacobi curve at λ of the corresponding extremal of g_1 to the Jacobi curve at $\Phi(\lambda)$ of the corresponding extremal g_2 (the curves are considered as unparametrized curves)

Theorem (I.Z.)

If a sub-Riemannian metric is not affinely rigid, then the Jacobi curve of a generic normal extremal is a direct product of curves in Lagrangian Grassmannians of smaller dimensions.

References

1. I. Zelenko, *On geodesic equivalence of Riemannian metrics and sub-Riemannian metrics on distributions of corank 1*, J. Math. Sci. (N. Y.) 135 (2006), no. 4, 3168-3194.

2. F. Jean, S. Maslovskaya, and I. Zelenko, *Inverse Optimal Control Problem: the Sub-Riemannian Case*, Proceedings of IFAC (International Federation of Automatic Control) IFAC-papersOnLine , vol 50, 2017, 7 pages.

3. F. Jean, S. Maslovskaya, and I. Zelenko, *On projective and affine equivalence of sub-Riemannian metrics*, Geom. Dedicata , volume 203, 279-319(2019).

4. F. Jean, S. Maslovskaya, and I. Zelenko, *On Weyl's type theorems and genericity of projective rigidity in sub-Riemannian Geometry,* preprint, submitted arXiv:2001.08584, 18 pages

5. A. Agrachev, I. Zelenko, Geometry of Jacobi curves. I,II , J. Dynamical and Control systems, 8(2002),No. 1, 93-140, No. 2, 167-215.

6. I. Zelenko, C. Li, *Differential geometry of curves in Lagrange Grassmannians with given Young diagram*, Differential Geometry and Its Applications, 27 (2009), 723-742.

THANK YOU VERY MUCH FOR YOUR ATTENTION!